Definitions: von Neumann entropy. In this series we will derive some useful properties of the von Neumann entropy: the quantum version of Shannon entropy.

The von Neumann entropy of a density operator $\rho \in \mathcal{S}(\mathscr{H}_A)$ is defined as

$$H(A)_\rho = -\operatorname{tr}\left(\rho \log \rho\right) = -\sum_i \lambda_i \log \lambda_i, \tag{1}$$

where $\{\lambda_i\}_i$ are the eigenvalues of ρ.

Given a composite system $\mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$ we write $H(AB)_\rho$ to denote the entropy of the reduced state of a subsystem, $\rho_{AB} = \operatorname{tr}_C(\rho_{ABC})$. When the state ρ is obvious from the context we drop the indices.

The *conditional* von Neumann entropy is defined as

$$H(A|B)_\rho = H(AB)_\rho - H(B)_\rho. \tag{2}$$

In the Alice-and-Bob picture this quantifies the uncertainty that Bob (who holds the B part of the quantum state ρ_{AB}) still has about Alice's state.

The strong sub-additivity property of the von Neumann entropy is very useful. It applies to a tripartite composite system $\mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$,

$$H(A|BC)_\rho \leq H(A|B)_\rho. \tag{3}$$

Exercise 1. *Properties of the von Neumann Entropy.*

(a) Prove the following general properties of the von Neumann entropy:

(i) $H(A)_\rho \geqslant 0$ for any ρ_A.

(ii) If ρ_{AB} is pure, then $H(A)_\rho = H(B)_\rho$.
Hint. Use the Schmidt decomposition of bipartite pure states: for any $|\psi\rangle_{AB}$, there exist coefficients p_k, and two orthonormal sets of vectors $\{|\chi_k\rangle_A\}_k$ and $\{|\phi_k\rangle_B\}_k$, such that $|\psi\rangle_{AB} = \sum_k \sqrt{p_k}\, |\chi_k\rangle_A \otimes |\phi_k\rangle_B$.

(iii) If two systems are independent, $\rho_{AB} = \rho_A \otimes \rho_B$, then $H(AB)_\rho = H(A)_{\rho_A} + H(B)_{\rho_B}$.

Solution.

(i) We have $H(A)_\rho = -\sum_k p_k \log p_k$, where p_k are the eigenvalues of ρ_A. But $-\log p_k$ is positive since probabilities are less than one, hence $H(A)_\rho \geqslant 0$.

(ii) This becomes clear when you apply the Schmidt decomposition to the pure state ρ_{AB}: the reduced states of the two subsystems A and B have the same eigenvalues and therefore the same von Neumann entropy.

(iii) We denote by $\{\lambda_i\}_i$ and $\{\gamma_j\}_j$ the eigenvalues of ρ_A and ρ_B respectively. Hence $\{\lambda_i\gamma_j\}_{i,j}$ are the eigenvalues of ρ_{AB} and we can write:

$$\begin{aligned} H(AB)_\rho &= -\sum_{i,j} \lambda_i\gamma_j \log(\lambda_i\gamma_j) \\ &= -\underbrace{\Big(\sum_i \lambda_i\Big)}_{=1} \cdot \Big(\sum_j \gamma_j \log \gamma_j\Big) - \underbrace{\Big(\sum_j \gamma_j\Big)}_{=1} \cdot \Big(\sum_i \lambda_i \log \lambda_i\Big) \\ &= H(A)_{\rho_A} + H(B)_{\rho_B}. \end{aligned}$$

(b) Consider a bipartite state that is classical on subsystem Z: $\rho_{ZA} = \sum_z p_z |z\rangle\langle z|_Z \otimes \rho_A^z$ for some basis $\{|z\rangle_Z\}_z$ of $\mathscr{H}_Z$. Show that:

(i) The conditional entropy of the quantum part, A, given the classical information Z is

$$H(A|Z)_\rho = \sum_z p_z \, H(A|Z = z), \tag{4}$$

where $H(A|Z = z) = H(A)_{\rho_A^z}$.

Solution. First, note that the eigenvalues of $\sum_z p_z |z\rangle\langle z| \otimes \rho_A^z$ are given by $\{p_z \lambda_k^z\}_{z,k}$, where $\{\lambda_k^z\}_k$ are the eigenvalues of $\rho_A^z \equiv \rho_{A|Z=z}$. We may now write:

$$\begin{aligned} H(AZ)_\rho &= -\sum_{z,k} p_z \lambda_k^z \log(p_z \lambda_k^z) \\ &= -\sum_z p_z \underbrace{\Big(\sum_k \lambda_k^z\Big)}_{=1} \log p_z - \sum_z p_z \Big(\sum_k \lambda_k^z \log \lambda_k^z\Big) \\ &= H(Z) + \sum_z p_z \, H(A|Z = z), \end{aligned}$$

and

$$H(A|Z)_\rho = H(AZ)_\rho - H(Z)_\rho = \sum_z p_z \, H(A|Z = z) \ .$$

(ii) The entropy of A is concave,

$$H(A)_\rho \geq \sum_z p_z \, H(A|Z = z). \tag{5}$$

Solution. First note that from strong sub-additivity follows sub-additivity, $H(AC) \leq H(A) + H(C)$, if $\mathscr{H}_B$ is empty. Applying this to a system classical in $\mathscr{H}_Z$, we get

$$H(AZ) \leqslant H(A) + H(Z) \ . \tag{S.1}$$

However, we also have as seen before

$$H(AZ) = H(Z) + \sum_z p_z \ H(A|Z = z) \ , \tag{S.2}$$

from which the inequality follows immediately.

(iii) The entropy of a classical probability distribution $\{p_z\}_z$ cannot be negative, even if one has access to extra quantum information, A,

$$H(Z|A)_\rho \geq 0. \tag{6}$$

Solution. Let us introduce a copy of the classical subsystem Z, Y, as follows:

$$\rho_{AZY} = \sum_z p_z |z\rangle\langle z|_Z \otimes |z\rangle\langle z|_Y \otimes \rho_A^z.$$

Note that, for this state, $H(AZ) = H(AY) = H(AZY)$.

We may now appply the strong sub-additivity,

$$
\begin{aligned}
& H(Y|AZ) \leq H(Y|A) \\
\Leftrightarrow\ & H(AZY) + H(A) \leq H(AZ) + \underbrace{H(AY)}_{=H(AZY)} \\
\Leftrightarrow\ & 0 \leq H(AZ) - H(A) \\
\Leftrightarrow\ & 0 \leq H(Z|A)
\end{aligned}
$$

Remark: Eq (6) holds in general only for classical Z. Bell states are immediate counterexamples in the fully quantum case.

Exercise 2. ***Von Neumann Entropy and Entanglement.***

(a) Compute the entropies $H(A)$, $H(AB)$ and the conditional entropy $H(A|B)$ of the Bell state

$$
|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|00\rangle_{AB} + |11\rangle_{AB}\right) \ ; \tag{7}
$$

(b) Calculate the conditional entropies $H(A|BC)$, $H(AB|C)$ and $H(A|B)$ of the GHZ state

$$
|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}} \left(|000\rangle_{ABC} + |111\rangle_{ABC}\right) \ . \tag{8}
$$

Solution.

(a) The reduced state on A is the fully mixed state, $\frac{1}{2}\mathbb{1}$. Then $H(A) = 1$, and $H(AB) = 0$ because the global state is pure. Then

$$
H(A|B) = H(AB) - H(A) = -1 \ . \tag{S.3}
$$

(b) The reduced state on A is fully mixed, $\frac{1}{2}\mathbb{1}$, and the reduced state on AB is classically correlated, $\rho_{AB} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$. Then

$$
H(A) = 1 \quad ; \quad H(AB) = 1 \quad ; \quad H(A|B) = 0 \ . \tag{S.4}
$$

Then since the global state is pure, $H(ABC) = 0$ and

$$
H(A|BC) = -1 \quad ; \quad H(AB|C) = -1 \ . \tag{S.5}
$$

Consider a separable state ρ_{AB}, i.e. a state that can be written as a convex combination of product states:

$$
\rho_{AB} = \sum_k p_k \, \rho_A^{(k)} \otimes \rho_B^{(k)} \ , \tag{9}
$$

where $\{p_k\}_k$ is a probability distribution.

(c) Prove that the von Neumann entropy is always positive for such a state,

$$
H\left(A|B\right)_\rho \geqslant 0 \ . \tag{10}
$$

Remark: This means that, whenever the conditional entropy is negative, you are necessarily in possession of an entangled state.

Hint. *First use the results of point (b) of the previous exercise to prove that the conditional von Neumann entropy is concave, i.e. if $\rho_{AB} = \sum_k p_k\, \rho^{(k)}_{AB}$, then*

$$H\left(A|B\right)_\rho \geqslant \sum_k p_k\, H\left(A|B\right)_{\rho^{(k)}} \ . \tag{11}$$

Solution. Let's first prove the claim, given in the hint, that the conditional von Neumann entropy is concave. Let $\rho_{AB} = \sum_k p_k\, \rho^{(k)}_{AB}$. As a convenience, let's introduce an extra classical system Z and define the state

$$\rho_{ABZ} = \sum_k p_k\, |k\rangle\langle k|_Z \otimes \rho^{(k)}_{AB} \ , \tag{S.6}$$

i.e. Z is an additional, fictive, register that contains the information about which of the product states $\rho^{(k)}_{AB}$ the system is actually in. Note that tracing out Z, we obtain the initial given state ρ_{AB}.

Consider the conditional entropy $H\left(A|B\right)_\rho$. By strong subadditivity, and writing out the entropies, we have

$$H\left(A|B\right)_\rho \geqslant H\left(A|BZ\right)_\rho = H\left(ABZ\right) - H\left(BZ\right) = H\left(AB|Z\right) - H\left(B|Z\right) \ . \tag{S.7}$$

We then use point (b) (i) of the previous exercise to write

$$\begin{aligned}(\text{S.7}) = \sum_k p_k \left(H\left(AB|Z=z\right)_\rho - H\left(B|Z=z\right)_\rho\right) &= \sum_k p_k \left(H\left(AB\right)_{\rho^{(k)}} - H\left(B\right)_{\rho^{(k)}}\right) \\ &= \sum_k p_k\, H\left(A|B\right)_{\rho^{(k)}} \ . \end{aligned} \tag{S.8}$$

Now, return to the main problem of the exercise, and let ρ_{AB} be a separable state of the form (9). By concavity of the conditional von Neumann entropy shown above, and using its additivity for independent systems,

$$\begin{aligned} H\left(A|B\right)_\rho \geqslant \sum_k p_k\, H\left(A|B\right)_{\rho^{(k)}_A \otimes \rho^{(k)}_B} &= \sum_k p_k \left[H\left(AB\right)_{\rho^{(k)}_A \otimes \rho^{(k)}_B} - H\left(B\right)_{\rho^{(k)}_B}\right] \\ &= \sum_k p_k \left[H\left(A\right)_{\rho^{(k)}_A} + H\left(B\right)_{\rho^{(k)}_B} - H\left(B\right)_{\rho^{(k)}_B}\right] = \sum_k p_k\, H\left(A\right)_{\rho^{(k)}_A} \geqslant 0 \ . \end{aligned}$$