Exercise 1. ***Bell-type Experiment.***

Consider a 2-qubit Hilbert space $\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$ with basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ in the Bell state

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \right). \tag{1}$$

Two parties, Alice and Bob, get half of state $|\psi^+\rangle$ so that Alice has qubit A and Bob has qubit B. The POVM corresponding to a measurement can be written in function of the angle α that the measurement basis makes with the $\{|0\rangle, |1\rangle\}$ basis,

$$M^\alpha = \left\{ |\alpha\rangle\langle\alpha|, |\alpha^\perp\rangle\langle\alpha^\perp| \right\}, \quad |\alpha\rangle = \cos\frac{\alpha}{2}\ |0\rangle + \sin\frac{\alpha}{2}\ |1\rangle, \quad |\alpha^\perp\rangle = -\sin\frac{\alpha}{2}\ |0\rangle + \cos\frac{\alpha}{2}\ |1\rangle,$$

where the 1/2 factor comes from the Bloch sphere notation. We label the outcomes $+$ for $|\alpha\rangle$ and $-$ for $|\alpha^\perp\rangle$.

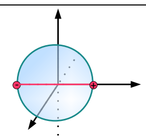
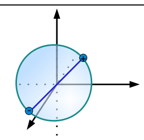
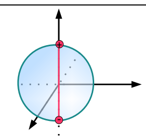
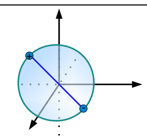
Suppose Alice performs such a measurement $\mathcal{M}^\alpha$ on her qubit.

(a) Find the description Bob would give to his partial state on B after he knows that Alice performed the measurement M_A^α on A. What description would Alice give to ρ_B given that she knows what measurement outcome she received?

(b) If Bob does the measurement $\mathcal{M}_B^0 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on B, what is the probability distribution for his outcomes, $\Pr_B$? How would Alice describe his probability distribution, $\Pr_{B|A}$?

(c) In part a) and b) Alice and Bob have different descriptions of the quantum state ρ_B and probability distribution of measurement outcomes on that state. Explain how this subjective assignment of the scenarios at B does not contradict with the actual measurement outcomes Bob will get after doing the measurement $\mathcal{M}_B^0$.

From now on look at the case where Alice and Bob can choose two different bases each:

meas.	0 	1 	2 	3 
Alice	$\alpha = 0, \begin{cases} \|0\rangle & \to + \\ \|\pi\rangle & \to - \end{cases}$		$\alpha = \frac{\pi}{2}, \begin{cases} \|\frac{\pi}{2}\rangle & \to + \\ \|\frac{-\pi}{2}\rangle & \to - \end{cases}$	
Bob		$\alpha = \frac{\pi}{4}, \begin{cases} \|\frac{\pi}{4}\rangle & \to + \\ \|\frac{5\pi}{4}\rangle & \to - \end{cases}$		$\alpha = \frac{3\pi}{4}, \begin{cases} \|\frac{3\pi}{4}\rangle & \to + \\ \|\frac{-\pi}{4}\rangle & \to - \end{cases}$

(d) The joint probabilities $P_{XY|ab}(x,y)$ of Alice and Bob obtaining outcomes x and y when they measure $A = a$ and $B = b$ are given in the following table.

Alice / Bob		A =0 +	A =0 −	A =2 +	A =2 −
B=1	+	$\frac{1}{2}-\epsilon$	ϵ	$\frac{1}{2}-\epsilon$	ϵ
	−	ϵ	$\frac{1}{2}-\epsilon$	ϵ	$\frac{1}{2}-\epsilon$
B=3	+	ϵ	$\frac{1}{2}-\epsilon$	$\frac{1}{2}-\epsilon$	ϵ
	−	$\frac{1}{2}-\epsilon$	ϵ	ϵ	$\frac{1}{2}-\epsilon$

with $\epsilon = \frac{1}{2}\sin^2(\pi/8) \approx 0.07$.

Compute

$$I_N(P_{XY|AB}) = P(X = Y|A = 0, B = 3) + \sum_{|a-b|=1} P(X \neq Y|A = a, B = b).$$

This quantity, similar to a Bell inequality, captures non-locality of quantum correlations: classically, it is at least equal to 1, whereas for quantum correlations it can be smaller than 1.

(e) Correlations of the above form that exist within quantum theory cannot be created classically. However, they are not the most general distributions we could consider if we are only contained by the no-signalling principle: there are in fact other joint distributions that cannot be obtained by measurements on a quantum state, but that nonetheless would not allow for instantaneous information transmission over distance (signalling). To see this, look at the following joint probability distribution, a so-called PR box:

Alice / Bob		A =0 +	A =0 −	A =2 +	A =2 −
B=1	+	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	−	0	$\frac{1}{2}$	0	$\frac{1}{2}$
B=3	+	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	−	$\frac{1}{2}$	0	0	$\frac{1}{2}$

Show that the PR box

(i) is non-signalling: $P(X|a, b_1) = P(X|a, b_2), \forall a$;

(ii) is non-local: $P_{XY|ab} \neq P_{X|a}P_{Y|b}$;

(iii) yields $I_N(P_{XY|AB}) = 0$.

(f) We shall now see how the above quantum correlation (coming from the Bell state) can be simulated using such a PR box combined with deterministic strategies. Imagine that Alice and Bob are given:

- with probability $1 - p$ a PR-box;
- with probability $p/4$, one of four deterministic boxes, that always outcome $++$, $+-$, $-+$ and $--$ respectively.

Find p so that the final joint probability distribution equals the one of the Bell state given above.

Solution.

(a) Let $P_\alpha = |\alpha\rangle\langle\alpha| \otimes \mathbb{1}_B$ and $P_{\alpha^\perp} = |\alpha^\perp\rangle\langle\alpha^\perp| \otimes \mathbb{1}_B$.
We have

$$|\alpha\rangle = \cos\frac{\alpha}{2}|0\rangle + \sin\frac{\alpha}{2}|1\rangle \qquad |\alpha\rangle\langle\alpha| = \begin{pmatrix} \cos^2\frac{\alpha}{2} & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix}$$

$$P_\alpha = \begin{pmatrix} \cos^2\frac{\alpha}{2} & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 \\ 0 & \cos^2\frac{\alpha}{2} & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & \sin^2\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & \sin^2\frac{\alpha}{2} \end{pmatrix}$$

$$\begin{aligned} |\alpha^\perp\rangle &= \cos\left(\frac{\alpha}{2}+\frac{\pi}{2}\right)|0\rangle + \sin\left(\frac{\alpha}{2}+\frac{\pi}{2}\right)|1\rangle \\ &= -\sin\frac{\alpha}{2}|0\rangle + \cos\frac{\alpha}{2}|1\rangle \end{aligned} \qquad |\alpha^\perp\rangle\langle\alpha^\perp| = \begin{pmatrix} \cos^2\frac{\alpha}{2} & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix}$$

$$P_{\alpha^\perp} = \begin{pmatrix} \sin^2\frac{\alpha}{2} & 0 & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 \\ 0 & \sin^2\frac{\alpha}{2} & 0 & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & \cos^2\frac{\alpha}{2} & 0 \\ 0 & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & \cos^2\frac{\alpha}{2} \end{pmatrix},$$

and the state shared by Alice and Bob is

$$|\Psi^+\rangle\langle\Psi^+| = \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The probability that Alice obtains outcome α is

$$\Pr_A(\alpha) = \operatorname{tr}\left[P_\alpha|\psi^+\rangle\langle\psi^+|\right] = \operatorname{tr}\left[\frac{1}{2}\begin{pmatrix} \cos^2\frac{\alpha}{2} & 0 & 0 & \cos^2\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \sin^2\frac{\alpha}{2} & 0 & 0 & \sin^2\frac{\alpha}{2} \end{pmatrix}\right] = \frac{1}{2}$$

and the reduced state on Bob's side after the measurement, from the point of view of Alice (who knows that the outcome of her measurement was α) is

$$\rho_{B|A=\alpha} = \operatorname{tr}_A\left(\frac{P_\alpha|\psi^+\rangle\langle\psi^+|}{\Pr_A(\alpha)}\right) = \operatorname{tr}_A\begin{pmatrix} \cos^2\frac{\alpha}{2} & 0 & 0 & \cos^2\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \sin^2\frac{\alpha}{2} & 0 & 0 & \sin^2\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\alpha}{2} & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix}.$$

Similarly, for the the other outcome, $\alpha^\perp$, we have

$$\Pr_A(\alpha^\perp) = \frac{1}{2} \qquad \rho_{B|A=\alpha^\perp} = \begin{pmatrix} \sin^2\frac{\alpha}{2} & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \cos^2\frac{\alpha}{2} \end{pmatrix}.$$

If Bob does not know that Alice performed a measurement, he sees his state as $\rho_B = \operatorname{tr}_A|\phi^+\rangle\langle\phi^+| = \frac{1}{2}\mathbb{1}_B$. Imagine now that Bob knows that Alice performed this measurement but does not know the outcome. All he know is that he has state $\rho_{B|A=\alpha}$ if she got α and state $\rho_{B|A=\alpha^\perp}$ if she got $\alpha^\perp$,

$$\begin{aligned} \rho_{B|A=?} &= \Pr_A(\alpha)\rho_{B|A=\alpha} + \Pr_A(\alpha^\perp)\rho_{B|A=\alpha^\perp} \\ &= \frac{1}{2}\begin{pmatrix} \cos^2\frac{\alpha}{2} & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \sin^2\frac{\alpha}{2} & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \cos^2\frac{\alpha}{2} \end{pmatrix} = \frac{\mathbb{1}_B}{2}. \end{aligned}$$

From Bob's viewpoint, it is the same whether Alice does not measure her qubit or measures it but does not tell him the outcome.

(b) We can read the probability distribution of $\mathcal{M}_B^0$ conditioned on A directly from our results in part (a),

$$\Pr_{B|A=\alpha}(x) = \begin{cases} \cos^2\frac{\alpha}{2}, & \text{for } x = 0 \\ \sin^2\frac{\alpha}{2}, & \text{for } x = 1 \end{cases}$$

For $A = \alpha^\perp$ the probabilities are interchanged. P_B (not conditioned on A) is just the uniform distribution over $0, 1$.

(c) The qubit of system B is correlated to another system (A) and what we are looking at are the states (and probability distributions) conditioned / not conditioned on an event on that system (measurement on A) that is itself random. More detailed analysis in the tips.

(d) The joint probabilities $P_{XY|ab}(x,y)$ of them obtaining outcomes x and y when they measure $A=a$ and $B=b$ are given by:

Alice / Bob		A =0, +	A =0, −	A =2, +	A =2, −
B=1	+	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$	ε
	−	ε	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$
B=3	+	ε	$\frac{1}{2}-\varepsilon$	$\frac{1}{2}-\varepsilon$	ε
	−	$\frac{1}{2}-\varepsilon$	ε	ε	$\frac{1}{2}-\varepsilon$

with $\varepsilon = \frac{1}{2}\sin^2(\pi/8) \approx 0.07$.

$$I_N(P_{XY|AB}) = \sum_{k=+,-}\left[P_{XY|A=0,B=3}(k,k) + \sum_{|a-b|=1} P_{XY|A=a,B=b}(k,\bar{k})\right]$$
$$= \sum \text{red terms in the table} = 8\varepsilon$$

(e) The PR-box is given by the following joint probability distribution:

Alice / Bob		A =0, +	A =0, −	A =2, +	A =2, −
B=1	+	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	−	0	$\frac{1}{2}$	0	$\frac{1}{2}$
B=3	+	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	−	$\frac{1}{2}$	0	0	$\frac{1}{2}$

The PR-box is

(i) *non-signalling:*

$$P_{X|A=a,B=1}(x) = P_{X|A=a,B=3}(x),\ \forall a,x \Leftrightarrow$$
$$\Leftrightarrow \sum_y P_{XY|A=a,B=1}(x,y) = \sum_y P_{X|A=a,B=3}(x,y),\ \forall a,x \Leftrightarrow$$
$$\Leftrightarrow \sum \text{red terms} = \sum \text{orange terms},\ \forall \text{ columns} \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{2} = \frac{1}{2}\ \checkmark$$

The other non-signalling condition, $P(Y|a_1,b) = P(Y|a_2,b), \forall b$, can be checked similarly, summing over rows instead of columns.

(ii) *non-local:*
Any local distribution is a convex combination of deterministic local distributions,

$$v$$

There are only 16 non-signalling deterministic local distributions:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right], \quad \text{etc.}$$

However, if we try to decompose the PR-box in this way we obtain:

$$\left[\begin{array}{cc|cc} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right] = \frac{1}{2}\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] + \frac{1}{2}\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$P_{XY|A=a,B=b}(x,y) \neq P_{X|A=a}(x)\ P_{Y|B=b}(y),\ \forall a,b,x,y \Leftrightarrow$$
$$\Leftrightarrow P_{XY|A=a,B=b}(x,y) \neq \left[\sum_{y',b'} P_{XY|A=a,B=b'}(x,y')\right]\left[\sum_{x',a'} P_{XY|A=a',B=b}(x',y)\right],\ \forall a,b,x,y \Leftrightarrow$$
$$\Leftrightarrow \{\text{table cell}\} \neq \left[\sum \{\text{column of the cell}\}\right]\left[\sum \text{row of the cell}\right],\ \forall \text{ cells} \Leftrightarrow$$
$$\Leftrightarrow 0 \text{ or } \frac{1}{2} \neq \left[\frac{1}{2}+\frac{1}{2}\right]\left[\frac{1}{2}+\frac{1}{2}\right] = 1\ \checkmark$$

(iii) *and yields* $I_N(P_{XY|AB}) = 0$.

Bob \ Alice		A =0 +	A =0 −	A =2 +	A =2 −
B=1	+	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	−	0	$\frac{1}{2}$	0	$\frac{1}{2}$
B=3	+	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	−	$\frac{1}{2}$	0	0	$\frac{1}{2}$

$$I_N(P_{XY|AB}) = \sum \text{red terms} = 0$$

(f) If we look at the first entry in the table (top left), it is straightforward to see that we need

$$(1-p) * \frac{1}{2} + p * \frac{1}{4} = \frac{1}{2} - \epsilon, \tag{S.1}$$

which implies that $p = 4\epsilon$. One can easily verify that this result also works for the other entries in the table.

See also: J. S. Bell, Phys. Vol. 1, No. 3, pp. 195–200, 1964.