Exercise 1. ***Composing Systems: The Tensor Product.***

You have learned from quantum mechanics that the composition of two systems described by states $|\psi_A\rangle \in \mathscr{H}_A$ and $|\psi_B\rangle \in \mathscr{H}_B$ is described by a state in the tensor product space $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$. The tensor product space is defined by its basis elements: if $\{|\phi_A^i\rangle\}$ and $\{|\phi_B^j\rangle\}$ are bases of $\mathscr{H}_A$ and $\mathscr{H}_B$, respectively, then

$$\mathscr{H}_A \otimes \mathscr{H}_B = \operatorname{span}\left\{|\phi_A^i\rangle \otimes |\phi_B^j\rangle\right\} \ . \tag{1}$$

The tensor product satisfies the following basic properties:

$$\big(|\psi_A\rangle + |\psi'_A\rangle\big) \otimes |\psi_B\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\psi'_A\rangle \otimes |\psi_B\rangle \ ; \tag{2}$$
$$(\alpha|\psi_A\rangle) \otimes |\psi_B\rangle = \alpha \cdot \ |\psi_A\rangle \otimes |\psi_B\rangle \ , \tag{3}$$

and the same properties hold on the second term.

(a) Consider two qubits, $\mathscr{H}_A = \mathscr{H}_B = \mathbb{C}^2$, with respective bases $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$. The tensor product space admits the basis $\{|0_A\rangle\otimes|0_B\rangle, |0_A\rangle\otimes|1_B\rangle, |1_A\rangle\otimes|0_B\rangle, |1_A\rangle\otimes|1_B\rangle\}$. Write the state of each of the following systems in this basis.

(i) System A in state $|0_A\rangle$ and system B in state $|1_B\rangle$.

(ii) System A in state $\frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle)$ and system B in state $|1_B\rangle$.

(iii) System A in state $\frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle)$ and system B in state $\frac{1}{\sqrt{2}}(|0_B\rangle + |1_B\rangle)$.

Solution.

(i) The joint system is in the state $|0_A\rangle \otimes |1_B\rangle$.

(ii) The joint state is $\frac{1}{\sqrt{2}}(|0_A\rangle \otimes |1_B\rangle + |1_A\rangle \otimes |1_B\rangle)$.

(iii) The joint state is

$$\frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle) \otimes \frac{1}{\sqrt{2}}(|0_B\rangle + |1_B\rangle) = \frac{1}{2}(|0_A\rangle \otimes |0_B\rangle + |0_A\rangle \otimes |1_B\rangle + |1_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle) \ . \tag{S.1}$$

An important property of the tensor product space is that there are states in $\mathscr{H}_A \otimes \mathscr{H}_B$ which cannot themselves be written as a tensor product of states from each space, i.e. they cannot be written in the form $|\psi_A\rangle \otimes |\psi_B\rangle$.

(b) Consider two qubits, $\mathscr{H}_A = \mathscr{H}_B = \mathbb{C}^2$, with respective bases $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$. Consider the state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle) \ . \tag{4}$$

Show that this state vector cannot be written as a tensor product of two individual state vectors in $\mathscr{H}_A$ and $\mathscr{H}_B$.

Solution. Suppose the state $|\Phi^+\rangle$ could be written as the product of two vectors in each Hilbert space, $|\Phi^+\rangle_{AB} = |\phi\rangle_A \otimes |\psi\rangle_B$. Then both $|\phi\rangle$ and $|\psi\rangle$ could be decomposed into the canonical basis of each Hilbert space,

$$
\begin{aligned}
|\phi\rangle &= \alpha_0 \, |0_A\rangle + \alpha_1 \, |1_A\rangle \ ; \\
|\psi\rangle &= \beta_0 \, |0_B\rangle + \beta_1 \, |1_B\rangle \ .
\end{aligned}
$$

Then

$$
|\phi\rangle \otimes |\psi\rangle = \alpha_0\beta_0 |0_A\rangle \otimes |0_B\rangle + \alpha_0\beta_1 |0_A\rangle \otimes |1_B\rangle + \alpha_1\beta_0 |1_A\rangle \otimes |1_B\rangle + \alpha_1\beta_1 |1_A\rangle \otimes |1_B\rangle \ ,
$$

which in comparision to $|\Psi^+\rangle = \frac{1}{\sqrt{2}} \left(|0_A\rangle \otimes |0_A\rangle + |1_B\rangle \otimes |1_B\rangle\right)$ yields the following set of equations for $\alpha_{1/2}$ and $\beta_{1/2}$,

$$
\begin{aligned}
\alpha_0\beta_0 &= \frac{1}{\sqrt{2}} \ ; & \alpha_0\beta_1 &= 0 \ ; \\
\alpha_1\beta_1 &= \frac{1}{\sqrt{2}} \ ; & \alpha_1\beta_0 &= 0 \ .
\end{aligned}
$$

It is obvious that this system does not admit any solution. (i.e., $\alpha_0\beta_1 = 0$ implies either $\alpha_0 = 0$ or $\beta_1 = 0$, so the equations on the left cannot be both non-zero.)

(c) Show that the state given in (b) can be written as

$$
|\Phi^+\rangle = \frac{1}{\sqrt{2}} \left(|+_A\rangle \otimes |+_B\rangle + |-_A\rangle \otimes |-_B\rangle\right) \ , \tag{5}
$$

with $|\pm_A\rangle = \frac{1}{\sqrt{2}} \left(|0_A\rangle + |1_A\rangle\right)$

Solution. Simply write out

$$
\begin{aligned}
&\frac{1}{\sqrt{2}} \left(|+_A\rangle \otimes |+_B\rangle + |-_A\rangle \otimes |-_B\rangle\right) \\
&\qquad = \frac{1}{2\sqrt{2}} \big(|0_A\rangle \otimes |0_B\rangle + |0_A\rangle \otimes |1_B\rangle + |1_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle \\
&\qquad\qquad + |0_A\rangle \otimes |0_B\rangle - |0_A\rangle \otimes |1_B\rangle - |1_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle\big) \\
&\qquad = \frac{1}{\sqrt{2}} \left(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle\right) \ .
\end{aligned} \tag{S.2}
$$

(d) *(Extra question, with an introduction to tomography.)* We have shown that no state vector can appropriately describe the system A from point (b). However, it can be described by a density operator. Determine the density operator ρ for that system by considering explicitely the probabilities of the outcomes of the measurements in the bases $\{|0_A\rangle, |1_A\rangle\}$, $\{|+_A\rangle, |-_A\rangle\}$, and $\{|+i_A\rangle, |-i_A\rangle\}$ (where $|\pm i_A\rangle = \frac{1}{\sqrt{2}} \left(|0_A\rangle \pm i|1_A\rangle\right)$).

Hints. By "measuring in a specific basis", it is meant to measure an observable that is diagonal in that basis. Recall also that the probability for measuring the outcome $|\phi\rangle$ is given by $\langle\phi|\rho|\phi\rangle$.

Solution. If the system A is measured in the basis $\{|0_A\rangle, |1_A\rangle\}$, then we need to collapse the system onto $|0_A\rangle$ or $|1_A\rangle$:

$$
\langle 0_A| \, \frac{1}{\sqrt{2}} \left(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle\right) = \frac{1}{\sqrt{2}} |0_B\rangle \ , \tag{S.3}
$$

since $\langle 0_A | 0_A\rangle = 1$ and $\langle 0_A | 1_A\rangle = 0$. We also have similarly

$$
\langle 1_A| \, \frac{1}{\sqrt{2}} \left(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle\right) = \frac{1}{\sqrt{2}} |1_B\rangle \ . \tag{S.4}
$$

The measurement outcome probabilities are simply the square of the norm of the resulting vector, which is $1/2$ each. In this case, they have to be $1/2$ anyway by symmetry of the two cases.

The considerations for measuring in the basis $|\pm_A\rangle$ are exactly the same, since we have seen in point (c) that the state takes the same form in the other basis.

The considerations for measuring in the basis $|\pm i_A\rangle$ are again exactly the same, as once can easily check.

Then the state, in the Bloch sphere representation, has to lie in the middle of all three axes. So it has to be the fully mixed state, given at the center of the Bloch sphere,

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \ . \tag{S.5}$$

Important Morale. The reduced state on one party of a fully entangled state is the fully mixed state.

In the density operator formalism, everything stays the same: the composition of two systems described by density operators $\rho_A \in \mathcal{S}(\mathscr{H}_A)$ and $\rho_B \in \mathcal{S}(\mathscr{H}_B)$ respectively, is described by a density operator $\rho \in \mathcal{S}(\mathscr{H}_A \otimes \mathscr{H}_B) = \mathcal{S}(\mathscr{H}_A) \otimes \mathcal{S}(\mathscr{H}_B)$. The important difference, however, is that ρ is not necessarily $\rho_A \otimes \rho_B$. Moreover, in contrast to state vectors, whatever the state of the joint system is, one can always write down the density operator of one part of the joint system, called *reduced state* or *marginal state*. The reduced state is obtained by *partial trace*.

(e) Write out the density operators for the following systems using basis elements of $\mathscr{H}_A \otimes \mathscr{H}_B$ given in point (a). (Use the matrix notation for convenience.)

(i) Two qubits in the state $|\Phi^+\rangle$ defined in point (b).

(ii) Two qubits that are randomly prepared either jointly in state $|0_A\rangle \otimes |0_B\rangle$ or in the joint state $|1_A\rangle \otimes |1_B\rangle$, with probability $1/2$ each.

(iii) *(Greenberger-Horne-Zeilinger state or cat state.)* Three qubits A, B, and C in the state described by the vector

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}} \left(|0_A\rangle \otimes |0_B\rangle \otimes |0_C\rangle + |1_A\rangle \otimes |1_B\rangle \otimes |1_C\rangle \right) \ . \tag{6}$$

(iv) The N-qubit version of the GHZ state,

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}} \left(|0_1\rangle \otimes \cdots \otimes |0_N\rangle + |1_1\rangle \otimes \cdots \otimes |1_N\rangle \right) \ . \tag{7}$$

(v) The maximally entangled state between two systems A and B, of N qubits each. Let $\{|i_{A/B}\rangle\}_i$ be a basis for each system. The state is given by

$$|\Psi_N\rangle = \frac{1}{\sqrt{2^N}} \sum_i |i_A\rangle |i_B\rangle \ . \tag{8}$$

Solution. The solution to this exercise is attached on page 5.

(f) Calculate the following reduced states from point (e) and give their density operators.

(1.) The reduced state of system (i) on qubit A (respectively on qubit B).

(2.) The reduced state of system (ii) on qubit A (respectively on qubit B).

(3.) The reduced state of the GHZ state (iii) on the two first qubits, A and B.

(4.) The reduced state of the N-qubit GHZ state (iv) on all but the last qubit, i.e. just tracing out the N-th qubit.

(5.) The reduced state of the maximally entangled state $|\Psi_N\rangle$ of point (v) on party A.

Hint. Factorize the state vector cleverly.

(6.) The reduced state of the maximally entangled state $|\Psi_N\rangle$ on the k first qubits of A and B (i.e. tracing out the $N-k$ last qubits of A and B).

Solution. The solution to this exercise is attached on page 7.

Solution to Exercise 1.

[Notation :
$|0\rangle_A \equiv |0_A\rangle$
$|00\rangle_{AB} \equiv |0\rangle_A \otimes |0\rangle_B$
$\equiv |0\rangle_A |0\rangle_B$]

(e) We have the following density operators -

(i) Bell state $|\Phi^+\rangle$.

$$|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)\left(\langle 00|_{AB} + \langle 11|_{AB}\right)$$

$$= \begin{pmatrix} 1/2 & & & 1/2 \\ & 0 & & \\ & & 0 & \\ 1/2 & & & 1/2 \end{pmatrix}$$

(missing entries are zero)

we have used: $|00\rangle_{AB} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|01\rangle_{AB} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ etc.

$\langle 00|_{AB} = (1 \; 0 \; 0 \; 0)$ etc.

(ii) By definition of the density operator,

$$\rho = \frac{1}{2}|00\rangle\langle 00|_{AB} + \frac{1}{2}|11\rangle\langle 11|_{AB}$$

$$= \begin{pmatrix} 1/2 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 1/2 \end{pmatrix} = \mathrm{diag}\left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$

notice: no off-diagonal terms !

(iii) GHZ state.

$$|GHZ\rangle\langle GHZ| = \frac{1}{2}\left(|000\rangle_{ABC} + |111\rangle_{ABC}\right)\left(\quad h.c. \quad\right)$$

$$= \begin{pmatrix} 1/2 & & & & & & & 1/2 \\ & 0 & & & & & & \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ 1/2 & & & & & & & 1/2 \end{pmatrix}$$

in the basis $|000\rangle, |001\rangle, |010\rangle, \ldots$ etc.

(iv) GHZ on N qubits. Same system:

$$|GHZ_N\rangle\langle GHZ_N| = \frac{1}{2}\left(|00\ldots0\rangle + |11\ldots1\rangle\right)\left(\langle 0\ldots0| + \langle 1\ldots1|\right)$$

$$= \begin{pmatrix} 1/2 & 0 & & & 1/2 \\ & & \ddots & & \\ & & & 0 & \\ 1/2 & & & & 1/2 \end{pmatrix}$$

in the basis $|0\ldots0\rangle, |0\ldots01\rangle, |0\ldots10\rangle$, etc.

(v) Maximally entangled state on N qubits.

$$|\Psi_N\rangle\langle\Psi_N| = \frac{1}{2^N}\sum_{i,j} |i\rangle_A|i\rangle_B\langle j|_A\langle j|_B = \frac{1}{2^N}\left(|0\ldots0\rangle_A|0\ldots0\rangle_B + \ldots + |1\ldots1\rangle_A|1\ldots1\rangle_B\right)$$

$$\cdot\left(\text{h.c.} \quad \text{———} \quad\right)$$

$$= \begin{array}{c} \overbrace{\hphantom{\frac{1}{2^N}\ 0\ \ddots\ 0}}^{2^N} \\ \left(\begin{array}{ccc|ccc|c|ccc} \frac{1}{2^N} & & & \frac{1}{2^N} & & & \cdots & & & \frac{1}{2^N} \\ & 0 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & 0 & & & & & & & \\ \hline & & & 0 & & & \cdots & & & \\ \frac{1}{2^N} & & & & \frac{1}{2^N} & & \cdots & & & \frac{1}{2^N} \\ & & & & 0 & & & & & \\ & & & & & 0 & & & & \\ \hline \vdots & & & \vdots & & \vdots & \ddots & & & \\ \hline & & & & & & & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & 0 & \\ \frac{1}{2^N} & & & \frac{1}{2^N} & & & \cdots & & & 1/2^N \end{array}\right) \end{array}$$

(f) Reduced States.

(1.)
$$\mathrm{tr}_B \begin{pmatrix} \frac12 & & & \frac12 \\ & 0 & & \\ & & 0 & \\ \frac12 & & & \frac12 \end{pmatrix}_{A,B} = \begin{pmatrix} \frac12 & 0 \\ 0 & \frac12 \end{pmatrix} \quad \text{(fully mixed state)}$$

$$\mathrm{tr}_A \begin{pmatrix} \boxed{\begin{matrix} \frac12 & \\ & 0 \end{matrix}} & \begin{matrix} & \frac12 \\ & \end{matrix} \\ \begin{matrix} & \\ \frac12 & \end{matrix} & \boxed{\begin{matrix} 0 & \\ & \frac12 \end{matrix}} \end{pmatrix} = \begin{pmatrix} \frac12 & 0 \\ 0 & \frac12 \end{pmatrix} \quad \text{(same)}$$

This confirms our calculation from point (d)!

(2.)
$$\mathrm{tr}_B \begin{pmatrix} \frac12 & & & \\ & 0 & & \\ & & 0 & \\ & & & \frac12 \end{pmatrix} = \begin{pmatrix} \frac12 & 0 \\ 0 & \frac12 \end{pmatrix}$$

$$\mathrm{tr}_A \begin{pmatrix} \frac12 & & & \\ & 0 & & \\ & & 0 & \\ & & & \frac12 \end{pmatrix} = \begin{pmatrix} \frac12 & 0 \\ 0 & \frac12 \end{pmatrix}$$

also fully mixed states.

Notice.

Both states $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $\frac12 |00\rangle\langle 00| + \frac12 |11\rangle\langle 11|$ give the same reduced states on A & B.

$\Rightarrow$ Knowing the reduced states does not tell you in which global state you are!!

(3.)
$$\mathrm{tr}_C \left(\begin{array}{cc|cc} \tfrac12\ddots & \ddots & \ddots & \ddots\,\tfrac12 \\ \ddots & \ddots & \ddots & \ddots \\ \hline \ddots & \ddots & \ddots & \ddots \\ \tfrac12\ddots & \ddots & \ddots & \ddots\,\tfrac12 \end{array}\right)_{A,B,C} = \begin{pmatrix} \tfrac12 & & & \\ & 0 & & \\ & & 0 & \\ & & & \tfrac12 \end{pmatrix}_{A,B} = \text{two classically correlated qubits}$$

(4.) Same idea: taking the sum of the diagonals of all the inner 2x2 blocks: only the $\underset{\text{upper-left \& bottom-right}}{\text{corner}}$ ½'s remain.

$$\mathrm{tr}_N\, |GHZ_N\rangle\langle GHZ_N| = \begin{pmatrix} \tfrac12 & 0 & & \\ & \ddots & & \\ & & 0 & \\ & & & \tfrac12 \end{pmatrix} = \tfrac12 |0\ldots0\rangle\langle 0\ldots0| + \tfrac12 |1\ldots1\rangle\langle 1\ldots1|$$
$$= \text{classical correlations}$$

(5.) First, notice that $|\Psi_N\rangle = \underbrace{|\Psi_1\rangle \otimes |\Psi_1\rangle \otimes \ldots \otimes |\Psi_1\rangle}_{N}$

= N pairs of maximally entangled qubits:

$$|\Psi_N\rangle = \frac{1}{\sqrt{2^N}}\Big(|\underbrace{0\ldots0}_{N}\rangle|\underbrace{0\ldots0}_{N}\rangle + |0\ldots01\rangle|0\ldots01\rangle + |0\ldots10\rangle|0\ldots10\rangle + \ldots + |1\ldots1\rangle|1\ldots1\rangle\Big)$$

$$= \frac{1}{\sqrt{2^N}}\Big[|0\rangle_{A_1}|0\rangle_{B_1} \otimes \big(|\underbrace{0\ldots0}_{N-1}\rangle|\underbrace{0\ldots0}_{N-1}\rangle + \ldots + |1\ldots1\rangle|1\ldots1\rangle\big)$$
$$+ |1\rangle_{A_1}|1\rangle_{B_1} \otimes \big(|0\ldots0\rangle|0\ldots0\rangle + \ldots + |1\ldots1\rangle|1\ldots1\rangle\big)\Big]$$

$$= \underbrace{\frac{1}{\sqrt2}\big(|00\rangle + |11\rangle\big)}_{=|\Phi^+\rangle = |\Psi_1\rangle} \otimes |\Psi_{N-1}\rangle \qquad \text{\& recursion.}$$

So: $\mathrm{tr}_B |\Psi_N\rangle\langle\Psi_N|$ = partial traces on B of all entangled qubit pairs

$$= \frac{\mathbb{1}_2}{2} \otimes \ldots \otimes \frac{\mathbb{1}_2}{2} = \frac{1}{2^N}\cdot \mathbb{1}_{2^N} \qquad \text{fully mixed state.}$$

(6.) The factorization given in point (5.) reads:

$$|\Psi_N\rangle_{AB} = \underbrace{|\Psi_1\rangle \otimes \ldots}_{\substack{k \text{ first} \\ \text{entangled pairs.}}} \otimes |\Psi_1\rangle$$

We are actually asked to trace out the first k entangled pairs of qubits (on both A and B).
But since $|\Psi_N\rangle$ is already factored as a product of pure states between what we're tracing out ($1 \ldots k$ pairs) and what we're keeping (pairs $k+1, \ldots, N$), we simply have

$$\mathrm{tr}_{1\ldots k} |\Psi_N\rangle\langle\Psi_N| = |\Psi_{(k+1)\ldots N}\rangle\langle\Psi_{(k+1)\ldots N}| \quad \left(\text{or: } |\Psi_{N-k}\rangle\langle\Psi_{N-k}|\right)$$

question of notation...

$= N-k$ maximally entangled qubits