Definitions: von Neumann entropy. In this series we will derive some useful properties of the von Neumann entropy: the quantum version of Shannon entropy.

The von Neumann entropy of a density operator $\rho \in \mathcal{S}(\mathscr{H}_A)$ is defined as

$$H(A)_\rho = -\operatorname{tr}\left(\rho \log \rho\right) = -\sum_i \lambda_i \log \lambda_i, \tag{1}$$

where $\{\lambda_i\}_i$ are the eigenvalues of ρ.

Given a composite system $\mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$ we write $H(AB)_\rho$ to denote the entropy of the reduced state of a subsystem, $\rho_{AB} = \operatorname{tr}_C(\rho_{ABC})$. When the state ρ is obvious from the context we drop the indices.

The *conditional* von Neumann entropy is defined as

$$H(A|B)_\rho = H(AB)_\rho - H(B)_\rho. \tag{2}$$

In the Alice-and-Bob picture this quantifies the uncertainty that Bob (who holds the B part of the quantum state ρ_{AB}) still has about Alice's state.

The strong sub-additivity property of the von Neumann entropy is very useful. It applies to a tripartite composite system $\mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$,

$$H(A|BC)_\rho \leq H(A|B)_\rho. \tag{3}$$

Exercise 1. ***Properties of the von Neumann Entropy.***

(a) Prove the following general properties of the von Neumann entropy:

(i) $H(A)_\rho \geqslant 0$ for any ρ_A.

(ii) If ρ_{AB} is pure, then $H(A)_\rho = H(B)_\rho$.
Hint. Use the Schmidt decomposition of bipartite pure states: for any $|\psi\rangle_{AB}$, there exist coefficients p_k, and two orthonormal sets of vectors $\{|\chi_k\rangle_A\}_k$ and $\{|\phi_k\rangle_B\}_k$, such that $|\psi\rangle_{AB} = \sum_k \sqrt{p_k}\,|\chi_k\rangle_A \otimes |\phi_k\rangle_B$.

(iii) If two systems are independent, $\rho_{AB} = \rho_A \otimes \rho_B$, then $H(AB)_\rho = H(A)_{\rho_A} + H(B)_{\rho_B}$.

(b) Consider a bipartite state that is classical on subsystem Z: $\rho_{ZA} = \sum_z p_z |z\rangle\langle z|_Z \otimes \rho_A^z$ for some basis $\{|z\rangle_Z\}_z$ of $\mathscr{H}_Z$. Show that:

(i) The conditional entropy of the quantum part, A, given the classical information Z is

$$H(A|Z)_\rho = \sum_z p_z\, H(A|Z=z), \tag{4}$$

where $H(A|Z=z) = H(A)_{\rho_A^z}$.

(ii) The entropy of A is concave,

$$H(A)_\rho \geq \sum_z p_z\, H(A|Z=z). \tag{5}$$

(iii) The entropy of a classical probability distribution $\{p_z\}_z$ cannot be negative, even if one has access to extra quantum information, A,

$$H(Z|A)_\rho \geq 0. \tag{6}$$

Remark: Eq (6) holds in general only for classical Z. Bell states are immediate counterexamples in the fully quantum case.

Exercise 2. ***Von Neumann Entropy and Entanglement.***

(a) Compute the entropies $H(A)$, $H(AB)$ and the conditional entropy $H(A|B)$ of the Bell state

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|00\rangle_{AB} + |11\rangle_{AB} \right) \ ; \tag{7}$$

(b) Calculate the conditional entropies $H(A|BC)$, $H(AB|C)$ and $H(A|B)$ of the GHZ state

$$|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}} \left(|000\rangle_{ABC} + |111\rangle_{ABC} \right) \ . \tag{8}$$

Consider a separable state ρ_{AB}, i.e. a state that can be written as a convex combination of product states:

$$\rho_{AB} = \sum_k p_k \, \rho_A^{(k)} \otimes \rho_B^{(k)} \ , \tag{9}$$

where $\{p_k\}_k$ is a probability distribution.

(c) Prove that the von Neumann entropy is always positive for such a state,

$$H\left(A|B\right)_\rho \geqslant 0 \ . \tag{10}$$

Remark: This means that, whenever the conditional entropy is negative, you are necessarily in possession of an entangled state.

Hint. First use the results of point (b) of the previous exercise to prove that the conditional von Neumann entropy is concave, i.e. if $\rho_{AB} = \sum_k p_k \, \rho_{AB}^{(k)}$, then

$$H\left(A|B\right)_\rho \geqslant \sum_k p_k \, H\left(A|B\right)_{\rho^{(k)}} \ . \tag{11}$$