Exercise 1. ***POVMs are the Most General Quantum to Classical Evolutions.***

To motivate why we consider POVMs in quantum information theory, we will show in this exercise that they capture the most general evolution of a quantum system into a classical register. A classical system X, in the quantum information formalism, is a quantum system (with Hilbert space $\mathscr{H}_X$) which is in a state ρ_X known to be diagonal in a fixed basis $\{|x\rangle\}$.

Let $\mathcal{E}_{A\to X} : \mathscr{L}(\mathscr{H}_A) \to \mathscr{L}(\mathscr{H}_X)$ be a trace-preserving, completely positive map from a (finite dimensional) quantum system A into a classical register X.

Show that this evolution is described by a POVM $\{A_x\}$ with the required properties, i.e. the probability of the (classical) output state to be in $|x\rangle\langle x|$ is $\operatorname{tr}(A_x\rho)$.

The following steps might help you, but it is not mandatory to follow them.

(a) Argue that $\mathcal{E}$ has to take the following form:

$$\mathcal{E}_{A\to X}(\rho) = \sum_x |x\rangle\langle x|\, f_x(\rho) \ , \tag{1}$$

where $f_x : \mathscr{L}(\mathscr{H}_A) \to \mathbb{R}$ is a linear mapping of ρ onto real numbers. $f_x(\rho)$ are the eigenvalues of $\mathcal{E}(\rho)$ in the eigenbasis $\{|x\rangle\}$ (which is fixed because, remember, X is a classical register).

(b) Argue that $f_x(\cdot)$ can be written in general as

$$f_x(\cdot) = \operatorname{tr}[\, A_x(\cdot)\,] \ , \tag{2}$$

for some hermitian operator A_x.

Hints. $\operatorname{tr}\left[A^\dagger B\right]$ *is the Hilbert-Schmidt scalar product in the space of linear operators* $\mathscr{L}(\mathscr{H}_A)$. *Also, what kind of object is* f_x*?*

(c) Argue that for all ρ, f_x has to take positive values and that the values for all x have to sum up to 1, $\sum_x f_x(\rho) = 1$. Deduce that $A_x \geqslant 0$ and $\sum_x A_x = \mathbb{1}$.

(d) Conclude from points (a)–(c).

Exercise 2. ***Distinguishing two quantum states***

Suppose you know the density operators of two quantum states $\rho, \sigma \in \mathscr{H}_A$. Then you are given one of the states at random—it may either be ρ or σ, with probability 1/2. The challenge is to perform a single measurement on your state and then guess which state that is.

(a) What is your best strategy? In which basis do you think you should perform the measurement? Can you express that measurement using a projector Q?

Hint. You can use the idea of exercise 1. What are you looking for? What should be the measurement outcome?

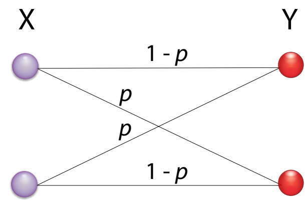
(b) Show that this optimal probability is directly related to the trace distance,

$$\Pr[\text{distinguish correctly}] = \frac{1}{2}\left[1 + \delta(\rho,\sigma)\right] \ . \tag{3}$$

Exercise 3. ***Classical channels as trace-preserving completely positive maps.***

In this exercise we will see how to represent classical channels as trace-preserving completely positive maps (TPCPMs).

(a) Take the binary symmetric channel $\mathbf{p}$,



Recall that we can represent the probability distributions on both ends of the channel as quantum states in a given basis: for instance, if $P_X(0) = q, P_X(1) = 1-q$, we may express this as the 1-qubit mixed state $\rho_X = q\ |0\rangle\langle 0| + (1-q)\ |1\rangle\langle 1|$.

What is the quantum state ρ_Y that represents the final probability distribution P_Y in the computational basis?

(b) Now we want to represent the channel as a map

$$\mathcal{E}_{\mathbf{p}} : \mathcal{S}(\mathscr{H}_X) \to \mathcal{S}(\mathscr{H}_Y)$$
$$\rho_X \mapsto \rho_Y.$$

An operator-sum representation (also called the Kraus-operator representation) of a CPTP map $\mathcal{E} : \mathcal{S}(\mathscr{H}_X) \to \mathcal{S}(\mathscr{H}_Y)$ is a decomposition $\{E_k\}_k$ of operators $E_k \in \mathrm{Hom}(\mathscr{H}_X, \mathscr{H}_Y)$, $\sum_k E_k {E_k}^\dagger = \mathbb{1}$, such that

$$\mathcal{E}(\rho_X) = \sum_k E_k \rho_X {E_k}^\dagger.$$

Find an operator-sum representation of $\mathcal{E}_{\mathbf{p}}$.

Hint. Think of each operator $E_k = E_{xy}$ as the representation of the branch that maps input x to output y.

(c) Now we have a representation of the classical channel in terms of the evolution of a quantum state. What happens if the initial state ρ_X is not diagonal in the computational basis?

(d) Now consider an arbitrary classical channel $\mathbf{p}$ from an n-bit space X to an m-bit space Y, defined by the conditional probabilities $\left\{P_{Y|X=x}(y)\right\}_{xy}$.

Express $\mathbf{p}$ as a map $\mathcal{E}_{\mathbf{p}} : \mathcal{S}(\mathscr{H}_X) \to \mathcal{S}(\mathscr{H}_Y)$ in the operator-sum representation.