Quantum Information Theory
Tips 11

Exercise 11.1 Resource inequalities: teleportation and classical communication

Exercise 11.2 A sufficient entanglement criterion

As stated in the question: "In general it is very hard to determine if a state is entangled or not." It's actually a very difficult problem that has had a lot of research effort invested to figure out how to determine if a state is entangled. Ideally, there would be a function that will give one outcome (say "1") if your state is entangled, and a different outcome (say "0") if it is not entangled. Unfortunately, no such easily calculable function exists, and so typically functions are used that are easy to calculate, but cannot identify all entangled states. Instead it is a function with two different outcomes: "1" there is entanglement, or "0" I don't know if the state is entangled or not. It's important to note that whenever entanglement is found by the function, it is and unambiguous answer.
In this exercise, we look at a particular kind of entanglement criterion called PPT. It is a function, such that if a state's partial transpose is positive, then it is not known whether the state is entangled or not. However, if the state's partial transpose is negative, then the state is entangled. It can be shown that for 2 by 2 dimensional systems (i.e. two qubits) or 2 by 3 dimensional systems, whenever you have a positive partial transpose, the state is definitely not entangled. We consider only these sized systems in this exercise.
Recall that we say that a bipartite state ρ_{AB} is separable (not entangled) if

$$\rho = \sum_k p_k \ \sigma_k \otimes \tau_k, \quad \forall k : p_k \geq 0, \sigma_k \in \mathcal{S}_=(\mathcal{H}_A), \tau_k \in \mathcal{S}_=(\mathcal{H}_B), \quad \sum_k p_k = 1.$$

Importantly, while there is this form for separable states, there is no such form, in general, for entangled states.

a) This is straightforward to calculate from the definition of a separable state. As stated in the question, it means that if you get a negative partial transpose, then the state must be entangled, because it could not be negative if the state was separable.

b) Here we introduce the transpose as a potential candidate, as it is positive, but not completely positive (and therefore we can get negative operators by applying the partial transpose).

c) You should find that for a certain range of ϵ, the state is entangled, while for another region it is not. This further illustrates why there is no easy way to tell if a state is entangled or not. For example, while it may look like

$$\frac{1}{4}|\psi^-\rangle\langle\psi^-| + \frac{3}{16}\mathbb{1}$$

is entangled, since it has some weight in a maximally entangled state, it turns out that this state is not actually entangled at all!

Exercise 11.3 Relative Entropy