Quantum Information Theory
Tips 8

Exercise 8.1 The Choi-Jamiołkowski Isomorphism

The CJ isomorphism gives a necessary and sufficient condition for CP: if

$$(\mathcal{E}_A \otimes \mathcal{I}_{A'})(|A||\Psi\rangle_{AA'}\langle\Psi|) \geq 0, \qquad |\Psi\rangle = \frac{1}{\sqrt{|A|}} \sum_i |i\rangle_A |i\rangle'_A,$$

then $\mathcal{E}_A$ is completely positive. The CJ isomorphism, τ, is the map from $\mathcal{E}_A$ to the state above. We will find an alternative way to derive τ.
First we define some notation. We use $|x\rangle\rangle$ to represent a non-normalised vector of a Hilbert space, just to keep in mind that it is not a quantum state. We do the same for non-normalised bras, $\langle\langle x|$. Now we represent operators as vectors. We can do this arbitrarily, as long as we know how to go back to the matrix representation. In this case we map the operators $C = \sum_{ij} c_{ij} |i\rangle_2 \langle j|_1$ as $|C\rangle\rangle = \sum_{ij} c_{ij} |i\rangle_2 |j\rangle_1$.

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}_{\mathcal{H}_1 \to \mathcal{H}_2} \longrightarrow \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \end{pmatrix}_{\mathcal{H}_2 \otimes \mathcal{H}_1}$$

Parts a) and b) are just to get tools that allows us to show why this would be interesting. It's just linear algebra.
In part a) we see how multiplying operators works in this picture. Here goes a little picture to keep track of all the different operators and Hilbert spaces.

$$\text{A: } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}_{\mathcal{H}_2 \to \mathcal{H}_3} \qquad \text{B: } \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}_{\mathcal{H}_1 \to \mathcal{H}_4} \qquad \text{C: } \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}_{\mathcal{H}_1 \to \mathcal{H}_2}$$

$$(\mathrm{ACB}^{\mathrm{T}})_{\mathcal{H}_4 \to \mathcal{H}_3}: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}_{\mathcal{H}_2 \to \mathcal{H}_3} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}_{\mathcal{H}_1 \to \mathcal{H}_2} \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \end{pmatrix}_{\mathcal{H}_4 \to \mathcal{H}_1}$$

In part b) we have only two operators A and B (here we write them as A and B, but they are written as X and Y on the exercise sheet), both from $\mathcal{H}_1$ to $\mathcal{H}_2$, and you have to show that $\mathrm{Tr}_1(|X\rangle\rangle\langle\langle Y|) = XY^*$. Here is another picture:

$$A: \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{\mathcal{H}_1 \to \mathcal{H}_2} \qquad B: \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}_{\mathcal{H}_1 \to \mathcal{H}_2} \qquad B^*: \begin{pmatrix} b^*_{11} & b^*_{21} \\ b^*_{12} & b^*_{22} \\ b^*_{13} & b^*_{23} \end{pmatrix}_{\mathcal{H}_2 \to \mathcal{H}_1}$$

$$\left(|A\rangle\rangle \langle\langle B| \right)_{\mathcal{H}_2\otimes\mathcal{H}_1 \to \mathcal{H}_2\otimes\mathcal{H}_1} : \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{pmatrix}_{\mathbb{C} \to \mathcal{H}_2\otimes\mathcal{H}_1} \begin{pmatrix} b^*_{11} & b^*_{12} & b^*_{13} & b^*_{21} & b^*_{22} & b^*_{23} \end{pmatrix}_{\mathcal{H}_2\otimes\mathcal{H}_1 \to \mathbb{C}}$$

Note: *the pictures are really just to give an idea of what is happening. Please stick to the nice ket/bra notation when solving the exercises.*

Now we will see how one can obtain the CJ isomorphism directly from the operator-sum representation of an operator. First, in part *c*), you use the results from *b*) and *a*) to show that we can obtain a matrix, T, from that representation. You should get $T = \sum_k |E_k\rangle\rangle\langle\langle E_k|$.

For part *c*) we have to prove that the T we obtained is the same from the CJ isomorphism. Hint: show that $d|\psi^+\rangle = |\mathbb{1}\rangle\rangle$.

Finally, check how to get the TP and CP conditions of a map directly from T. The script may help.

Another note: Sorry for the notation mess: first of all, X, Y and Z are in the pictures called A, B, and C.

Exercise 8.2 Measurements as unitary evolutions

In the script, CPTP maps of the type we consider here are discussed. In particular, the Kraus operators for a projective measurement are described as $|x\rangle \otimes P_x$. This is true, as it denotes the classical outcome x in the quantum state $|x\rangle$ and P_x are the projectors acting on the quantum state that is input. However, if you are not interested in the classical outcome, and are only interested in how the quantum state changes under this projective measurement, then you can just consider the Kraus operators as P_x. Please do so for this question to make it easier. If you want, you can also do this question by keeping track of the classical outcome register, you will still end up with the same conclusions.