ETH
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Quantum Information Theory
Series 11

HS 12
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Exercise 11.1 Resource inequalities: teleportation and classical communication

We saw a protocol, teleportation, to transmit one qubit using two bits of classical computation and one ebit, $\underset{\Upsilon}{\overset{2}{\rightarrow}} \geq \overset{1}{\rightsquigarrow}$ (Section 6.1 of the script). Now suppose that Alice and Bob share unlimited entanglement: they can use up as many ebits as they want. Can Alice send n qubits to Bob using less than $2n$ bits of classical communication? In other words, we want to know if the following is possible:

$$\underset{\underset{\infty}{\Upsilon}}{\overset{m}{\rightarrow}} \geq \underset{\underset{\infty}{\Upsilon}}{\overset{n}{\rightsquigarrow}}, \qquad m < 2n.$$

Prove that this is not the case. **Hint:** Use superdense coding.

Exercise 11.2 A sufficient entanglement criterion

In general it is very hard to determine if a state is entangled or not. In this exercise we will construct a simple entanglement criterion that correctly identifies all entangled states in low dimensions.
Recall that we say that a bipartite state ρ_{AB} is separable (not entangled) if

$$\rho = \sum_k p_k \ \sigma_k \otimes \tau_k, \quad \forall k : p_k \geq 0, \sigma_k \in \mathcal{S}_=(\mathcal{H}_A), \tau_k \in \mathcal{S}_=(\mathcal{H}_B), \quad \sum_k p_k = 1.$$

a) Let $\Lambda_A : \mathrm{End}(\mathcal{H}_A) \mapsto \mathrm{End}(\mathcal{H}_A)$ be a positive map. Show that $\Lambda_A \otimes \mathcal{I}_B$ maps separable states to positive operators. This means that if we apply $\Lambda_A \otimes \mathcal{I}_B$ to a bipartite state ρ_{AB} and obtain a non-positive operator, we know that ρ_{AB} is entangled. In other words, this is a sufficient criterion for entanglement.

b) Now we have to find a suitable map Λ_A. Show that the transpose,

$$\mathcal{T}\left(\sum_{ij} a_{ij} |i\rangle\langle j|\right) = \sum_{ij} a_{ji} \ |i\rangle\langle j|,$$

is a positive map from $\mathrm{End}(\mathcal{H}_A)$ to $\mathrm{End}(\mathcal{H}_A)$, but is not completely positive.

c) Apply the partial transpose, $\mathcal{T}_A \otimes \mathcal{I}_B$, to the ε-noisy Bell state

$$\rho^\varepsilon_{AB} = (1-\varepsilon) \ |\psi^-\rangle\langle\psi^-| + \varepsilon \ \frac{\mathbb{1}_4}{4}, \quad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad \varepsilon \in [0,1].$$

For what values of ε can we be sure that ρ^ε is entangled?

Remark: Indeed, it can be shown that the PPT criterion (positive partial transpose) is necessary and sufficient for systems of dimension 2×2 and 2×3.

Exercise 11.3 Relative Entropy

The quantum relative entropy is defined as $D(\rho||\sigma) = \mathrm{Tr}(\rho \log \rho - \rho \log \sigma)$. For two classical probability distributions p and q, this definition simplifies to the expression for the Kullback-Leibler divergence $\sum_j p_j \log \frac{p_j}{q_j}$. Similar to the classical case, the relative entropy serves as a kind of "distance" between quantum states (although it is not technically a metric). Show that

a) $H(A|B) = -D(\rho||\mathbb{1}_A \otimes \rho_B)$

b) $D(\rho||\sigma) \geq 0$, with equality if and only if $\rho = \sigma$

c) $D(\rho||\sigma) \leq \sum_k p_k D(\rho_k||\sigma)$, where $\rho = p_1\rho_1 + p_2\rho_2$

d) For any CPTPM $\mathcal{E}$, $D(\rho||\sigma) \geq D(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$

e) $D(\rho||\sigma)$ is not a metric. Show this by proving that it is not symmetric.