# Quantum Information Theory
# Series 5



### Exercise 5.1 Purification

A decomposition of a state $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ is a (non-unique) convex combination of pure states $\rho_A^x = |a_x\rangle\langle a_x|$ such that $\rho_A = \sum_x \lambda_x \rho_A^x$.

*a*) Show that $|\Psi\rangle = \sum_x \sqrt{\lambda_x}\, |a_x\rangle_A \otimes |b_x\rangle_B$ is a purification of $\rho_A$ for *any* orthonormal basis $\{\, |b_x\rangle_B \,\}_x$ of $\mathcal{H}_\mathrm{B}$.

*b*) Show that any two purifications are related by a *local* isometry on the purifying system.

*c*) Mixed states can be decomposed in many different ways, i.e. with respect to many different bases. We will show that, from a purification of a mixed state $\rho_A$, we can generate any decomposition $\{\rho_A^x\}_x$ such that $\rho_A = \sum_x \lambda_x \rho_A^x$ by performing measurements on the purifying system. This is sometimes called *steering*.

For $\rho_A$ as defined above, and any purification $|\Phi\rangle$ of $\rho_A$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, find a measurement on $\mathcal{H}_B$, described by operators $\{\, M_B^x \,\}_x$, such that

$$\lambda_x = \mathrm{Tr}\left[|\Phi\rangle\langle\Phi|(\mathbb{1}_A \otimes M_B^x)\right] \quad \text{and} \quad \rho_A^x = \frac{\mathrm{Tr}_B\left[|\Phi\rangle\langle\Phi|(\mathbb{1}_A \otimes M_B^x)\right]}{\lambda_x}. \tag{1}$$

In this picture $\lambda_x$ is the probability of measuring $x$ and $\rho_A^x$ is the state after the measurement.

[ **Decomposition example.** To see what we mean by the different compositions, look at the following example for the fully mixed state. It can be written as

$$\begin{aligned}\frac{\mathbb{1}_2}{2} &= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}, \quad \text{or} \\ &= \frac{|+\rangle\langle +| + |-\rangle\langle -|}{2}, \quad \text{or (more interestingly)} \\ &= \sum_{i=1}^{4} \frac{1}{4} |\theta_i, \phi_i\rangle\langle\theta_i, \phi_i|,\end{aligned}$$

where $\{|\theta_i, \phi_i\rangle\}$ are the pure states:

$$(\theta_1, \phi_1) = (\psi, \frac{\pi}{4}), \quad (\theta_2, \phi_2) = (\pi - \psi, -\frac{\pi}{4}), \quad (\theta_3, \phi_3) = (\pi - \psi, \frac{3\pi}{4}), \quad (\theta_4, \phi_4) = (\psi, -\frac{3\pi}{4}),$$

with $\psi := \arccos(\frac{1}{\sqrt{3}})$. Note that as these states are pure, they lie on the surface of the sphere, and therefore can be parametrized by just two parameters $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$. You can also see that the vectors sit on the vertices of a regular tetrahedron.

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### Exercise 5.2 Distinguishing two quantum states

Suppose you know the density operators of two quantum states $\rho, \sigma \in \mathcal{H}_A$. Then you are given one of the states at random—it may either be $\rho$, with probability $p$, or $\sigma$, with probability $1 - p$. The challenge is to perform a single measurement on your state and then guess which state that is.

*a*) What is your best strategy? In which basis do you think you should perform the measurement? Can you express that measurement using a projector $P$?

*b*) What is the probability of guessing correctly, $\mathrm{Pr}^p_\checkmark(\rho, \sigma)$? Compare that with the case where the states are evenly distributed, $\mathrm{Pr}^{0.5}_\checkmark(\rho, \sigma) = \frac{1}{2}[1 + \delta(\rho, \sigma)]$, where $\delta(\rho, \sigma)$ is the trace distance between the two quantum states.

**Exercise 5.3 Distance bounds**

Maximally entangled states $|\Psi\rangle$ between two systems $A$ and $A'$ are used for many quantum tasks (for instance teleportation, which we will introduce later in the lecture). Unfortunately, sometimes we cannot be 100% sure we can create exactly $|\Psi\rangle$. Suppose, however, that we do know how to create a state in $A \otimes B$ such that $B$ has almost no information about $A$: $\rho_{AB} \approx \mathbb{1}_A/|A| \otimes \rho_B$. As we shall see, this can help us find an (approximately) maximally entangled state between $A$ and $A'$.

*a)* Given a trace-preserving quantum operation $\mathcal{E}$ and two states $\rho$ and $\sigma$, show that

$$\delta\left(\mathcal{E}(\sigma), \mathcal{E}(\rho)\right) \leq \delta(\sigma, \rho).$$

How would you interprete this statement?

*b)* Show that any purification of the state $\rho_{AB} = \frac{\mathbb{1}_A}{|\mathcal{H}_A|} \otimes \rho_B$ has the form

$$|\psi\rangle_{AA'BB'} = |\Psi\rangle_{AA'} \otimes |\psi\rangle_{BB'},$$

where $|\Psi\rangle_{AA'} = |\mathcal{H}_A|^{-\frac{1}{2}} \sum_i |i\rangle_A |i\rangle_{A'}$ is a maximally entangled state, and $|\psi\rangle_{BB'}$ is a purification of $\rho_B$.

*c)* Show that $1 - F(\rho, \sigma) \leq \delta(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}$.

*d)* Consider a state $\sigma_{AB}$ that is $\varepsilon$-distant from $\rho_{AB}$ according to the trace distance, i.e.

$$\delta\left(\sigma_{AB}, \frac{\mathbb{1}_A}{|\mathcal{H}_A|} \otimes \rho_B\right) \leq \varepsilon.$$

Find an upper bound for

$$\delta\left(\tau_{AA'}, |\Psi\rangle_{AA'}\langle\Psi|_{AA'}\right),$$

where $|\phi\rangle_{AA'BB'}$ is a purification of $\sigma_{AB}$ and $\tau_{AA'} = \mathrm{Tr}_{BB'}(|\phi\rangle_{AA'BB'}\langle\phi|_{AA'BB'})$.