Quantum Field Theory I — Problem Set 10

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1. Linear sigma model with trivial vacuum

Consider a model of N real scalar fields Φ^i that couple to each other through a quartic interaction that is symmetric under $SO(N)$ rotations of the N fields. The Lagrangian of this model is

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\Phi^i\partial_\mu\Phi^i - \tfrac{1}{2}m^2\Phi^i\Phi^i - \tfrac{1}{8}\lambda\left(\Phi^i\Phi^i\right)^2. \tag{1}$$

a) Derive the corresponding Hamiltonian and show that

$$V(\Phi) = \tfrac{1}{2}m^2\Phi^i\Phi^i + \tfrac{1}{8}\lambda\left(\Phi^i\Phi^i\right)^2 \tag{2}$$

is the potential term of the Hamiltonian.

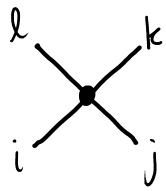
First consider the case $m^2 > 0$, convince yourself that for $\lambda = 0$ the Hamiltonian is just an N-fold copy of the Klein–Gordon Hamiltonian. For small λ we can calculate a perturbation series in λ.

b) Show that the Wick contraction of the Φ^i fields is

$$\underbrace{\Phi^i(x)\,\Phi^j}(y) = -i\delta^{ij}G_\mathrm{F}(x-y), \tag{3}$$

where G_F is the Feynman propagator for a Klein-Gordon scalar field of mass m.

c) Show that there is one interaction vertex given by



$$= -i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right). \tag{4}$$

d) Let $N = 2$ and compute at leading order in λ the differential cross section $d\sigma/d\Omega$ for

$$\Phi^1\Phi^1 \to \Phi^2\Phi^2, \tag{5}$$
$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \tag{6}$$
$$\Phi^1\Phi^1 \to \Phi^1\Phi^1. \tag{7}$$

Note that the differential cross section for four particles in the centre of mass frame is given by

$$\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2 s}, \tag{8}$$

where s is the centre of mass energy squared of the system and M is the matrix element describing the process.

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2. Linear sigma model with non-trivial vacuum

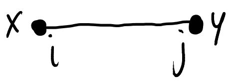
Next we consider the case where $m^2 =: -\mu^2 < 0$. Convince yourself that $V(\Phi)$ has a local maximum at $\Phi^i = 0$. As the potential is bounded from below, the minimum must be located at a non-vanishing value of Φ^i. Moreover, the theory is invariant under global $SO(N)$ rotations of the fields, and all points on the sphere with equal $|\Phi|$ must also be minima of $V(\Phi)$. The ground state of our field theory is therefore given by some non-zero constant field Φ^i. We choose Φ^i to point along the N-th direction or use a $SO(N)$ rotation to that end. We parametrise the quantum fields around the vacuum as

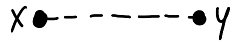
$$\Phi^i(x) = \phi^i(x), \qquad i = 1, \dots, N-1, \tag{9}$$
$$\Phi^N(x) = v + \sigma(x). \tag{10}$$

a) Determine the vacuum expectation value v, i.e. the field value in the minimum, in terms of μ and λ by minimising the potential $V(\Phi)$.

b) Insert the ansatz for Φ in terms of ϕ and σ and the expression for v into the Lagrangian (or Hamiltonian) and show that the new Lagrangian (Hamiltonian) describes a theory of a massive field σ and $N-1$ massless fields ϕ^i.

c) Convince yourself that the ϕ and σ fields interact through a new set of cubic and quartic vertices and determine the Feynman rules for all propagators and vertices.

$\phi^i(x)\,\phi^j(y) =$ 

$\sigma(x)\,\sigma(y) =$ 

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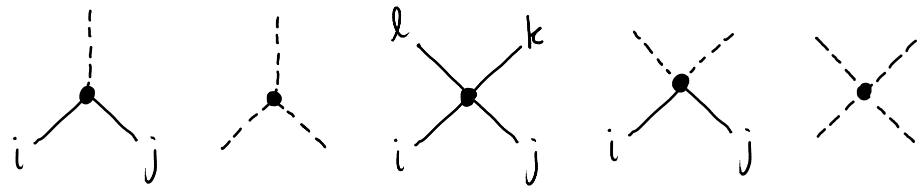

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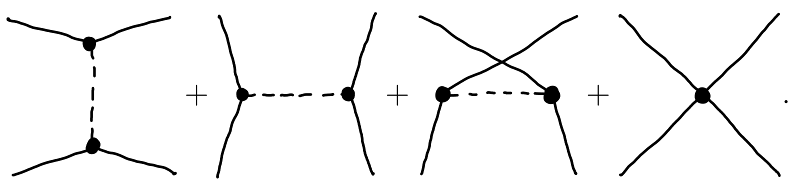
d) Let $N = 2$ and compute at leading order in λ the differential cross section $d\sigma/d\Omega$ for

$$\phi^1\phi^1 \to \phi^2\phi^2, \tag{13}$$
$$\phi^1\phi^2 \to \phi^1\phi^2, \tag{14}$$
$$\phi^1\phi^1 \to \phi^1\phi^1. \tag{15}$$

Note that there are now four Feynman diagrams contributing to the amplitude at leading order

$M =$  . (16)