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1. Spinor rotations

The Dirac equation is invariant under Lorentz transformations $\Psi'(x') = S\Psi(x)$ if the spinor transformation matrix S satisfies

$$\Lambda^\mu{}_\nu \, S^{-1}\gamma^\nu S = \gamma^\mu. \tag{1}$$

For an infinitesimal Lorentz transformation $\Lambda_{\mu\nu} = \eta_{\mu\nu} + \delta\omega_{\mu\nu}$ this is fulfilled if

$$S = 1 + \tfrac{1}{8}\delta\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]. \tag{2}$$

a) Find the infinitesimal spinor transformation δS for a rotation around the 3-axis, i.e. only $\delta\omega_{12} = -\delta\omega_{21} \neq 0$.

b) Finite transformations are obtained by considering a consecutive application of infinitely many, $N \to \infty$, infinitesimal transformations $\delta\omega = \omega/N$

$$S = \lim_{N\to\infty}\left(1 + \frac{1}{8}\frac{\omega_{\mu\nu}}{N}[\gamma^\mu, \gamma^\nu]\right)^N = \exp\left(\tfrac{1}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]\right). \tag{3}$$

Compute the finite rotation with angle ω_{12} around the same axis as before. Also compute the finite transformation $\Lambda = \exp(\omega)$ for vectors.

c) What happens to the individual components of a spinor under this transformation? What is the period of the transformation in the angle ω_{12}? Compare it to the finite rotation for vectors.

2. Completeness for gamma matrices

An arbitrary product of γ-matrices is proportional to one of the following 16 linearly independent matrices γ^a (here a is a multi-index which specifies the type of matrix, S, P, V, A, T, along the corresponding indices if any)

- $\Gamma^{\mathrm{S}} = 1$,
- $\Gamma^{\mathrm{P}} = \gamma^5$,
- $\Gamma^{\mathrm{V},\mu} = \gamma^\mu$,
- $\Gamma^{\mathrm{A},\mu} = i\gamma^5\gamma^\mu$,
- $\Gamma^{\mathrm{T},\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

a) Show that the trace of any product of Γ matrices is given by $\mathrm{tr}(\Gamma^a\Gamma^b) = 4\delta^{ab}$. For simplicity we ignore the signs arising from the Lorentz signature.

b) Show that for any $a \neq b$ there is a $n \neq \mathrm{S}$ such that $\Gamma^a\Gamma^b = \alpha\Gamma^n$ with some $\alpha \in \mathbb{C}$.

c) Show that the matrices are linearly independent and therefor form a complete basis of 4×4 spinor matrices. *Hint:* To do this consider a sum $\sum_a \alpha_a\Gamma^a = 0$. What can be said about the coefficients?

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3. Fierz identity

a) Use the linear independence of the Γ^a matrices to show that

$$\delta^\alpha_\gamma \delta^\beta_\delta = \sum_i \frac{1}{4} (\Gamma_i)^\alpha{}_\delta (\Gamma_i)^\beta{}_\gamma . \tag{4}$$

Hint: Decompose an arbitrary matrix $M^\alpha{}_\beta = \sum_i m_i (\Gamma^i)^\alpha{}_\beta$ and find the coefficients m_i.

b) Use the result from a) to show the Fierz identity:

$$(\Gamma^i)^\alpha{}_\beta (\Gamma^j)^\gamma{}_\delta = \sum_{k,l} \frac{1}{16} \operatorname{tr}(\Gamma^i \Gamma^l \Gamma^j \Gamma^k) (\Gamma^k)^\alpha{}_\delta (\Gamma^l)^\gamma{}_\beta . \tag{5}$$

c) Find the Fierz transformation for the spinor products

- $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$,
- $(\bar{u}_1 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_4)$.