11 Loop Corrections

Finally, we will discuss some basic loop effects and how to deal with the divergent loop integrals we shall encounter.

We use a scalar theory with a quartic potential

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 - \tfrac{1}{6}\kappa\mu\phi^3 - \tfrac{1}{24}\kappa^2\lambda\phi^4. \tag{11.1}$$

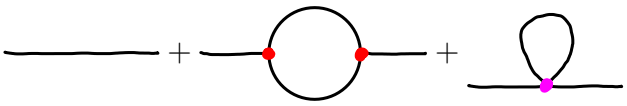
The mass is denoted by μ and there are two dimensionless constants κ and λ. We assume κ to be small and λ to be a freely tunable parameter.

11.1 Self Energy

First, we consider the time-ordered two-point correlator in order to isolate the pole associated to asymptotic particles. In momentum space we know by means of momentum conservation

$$F_2(p,q) = -i(2\pi)^4\delta^4(p+q)M_2(p). \tag{11.2}$$

Leading Orders. Let us evaluate the first few orders of the function M_2 from Feynman diagrams.



(11.3)

Obviously the leading contribution is the isolated Feynman propagator

$$M_2^{(0)}(p) = \frac{1}{p^2+\mu^2-i\epsilon}\,. \tag{11.4}$$

The first correction involves one loop in the diagram. The tadpole diagram can be safely ignored.[1] The bubble diagram amounts to the following loop integral[2]

$$\begin{aligned} M_2^{(2)}(p) &= \frac{i(-i\mu\kappa)^2(-i)^4}{(p^2+\mu^2-i\epsilon)^2}\, iI(-p^2)\,, \\ I(-p^2) &= \frac{1}{2}\int\frac{-id^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2+\mu^2-i\epsilon}\,\frac{1}{(p-\ell)^2+\mu^2-i\epsilon}\,. \end{aligned} \tag{11.5}$$

The resulting quadruple integral is difficult to perform, in particular due to two different denominators. We will postpone the evaluation, and discuss the implications for a generic function $I(-p^2)$.

[1] We shall see later on how this can be achieved in practice.

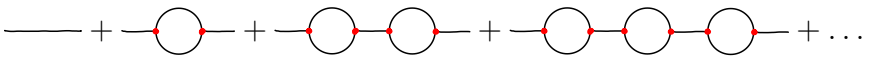
[2] The resulting integral is Lorentz invariant and can therefore be written as a function of p^2 (or $-p^2$).

Mass Shift. Altogether we obtain for the two-point function

$$M_2(p) = \frac{1}{p^2 + \mu^2 - i\epsilon} + \frac{\mu^2\kappa^2 I(-p^2)}{(p^2 + \mu^2 - i\epsilon)^2} + \dots . \tag{11.6}$$

Now this expression appears to have a double pole at $p^2 = -\mu^2$. From our earlier discussion we know that this should not happen; there should only be single poles in the two-point function.

Taking a peek at higher loop orders, we find, among others, a sequence of iterated loop integrals terms.

 (11.7)

All of these we can express easily in the form of a geometric series

$$M_2(p) = \frac{1}{p^2 + \mu^2 - i\epsilon} \sum_{k=0}^{\infty} \left(\frac{\mu^2\kappa^2 I(-p^2)}{p^2 + \mu^2 - i\epsilon} \right)^k + \dots . \tag{11.8}$$

Pushing convergence questions aside, we sum the series

$$M_2(p) = \frac{1}{p^2 + \mu^2 - \mu^2\kappa^2 I(-p^2) - i\epsilon} + \dots . \tag{11.9}$$

In other words we have moved the correction term to the denominator of the propagator.

Why should we include precisely these higher-order terms into our first order correction? There are several reasons:

- There is nothing wrong with it. We just have to make sure to eventually count every higher-order diagram with the correct weight.
- We avoid unwanted higher poles in the two-point function. The diagrams contributing to the denominator are one-particle irreducible. They cannot be cut into two parts by cutting a single internal line. Such diagrams are not expected to produce additional poles at the mass shell.
- The inverse of the two-point function at leading order is directly related to the action. It therefore appears somewhat more natural to expand the inverse $1/M_2(p)$ rather than the original function $M_2(p)$. Our correction terms are simply the first correction to $1/M_2(p)$.
- A mass counterterm (to be discussed below) yields the same result.
- In QFT2 we will introduce a useful functional that includes $1/M_2(p)$.

In this form we generically expect only single poles. Particularly, the pole which was originally at $p^2 + \mu^2 = 0$ may now have shifted to a new location $p^2 + m^2 = 0$. Assuming that this is the case, we can determine the new mass and also the residue at the pole. For the mass, we should solve the equation

$$-m^2 + \mu^2 - \mu^2\kappa^2 I(m^2) + \dots = 0. \tag{11.10}$$

The assumption of the perturbative treatment is that κ is small and that m is approximated well by μ. Hence we can replace $I(m^2)$ by $I(\mu^2)$ and therefore the new mass to leading order reads

$$m^2 = \mu^2 - \mu^2\kappa^2 I(\mu^2) + \dots . \tag{11.11}$$

How about the residue? The expansion of the denominator at $p^2 + m^2 = 0$ reads

$$(p^2 + m^2)\left(1 + \mu^2\kappa^2 I'(m^2)\right) + \dots . \tag{11.12}$$

Altogether the pole of the two-point function corresponding to the asymptotic particle takes the form

$$M_2(p) = \frac{Z}{p^2 + m^2 - i\epsilon} + \dots \tag{11.13}$$

with the field strength renormalisation

$$Z = 1 - \mu^2\kappa^2 I'(\mu^2) + \dots . \tag{11.14}$$

Spectral Function. We can now extract the spectral function $\rho(s)$ from $M_2(p)$

$$M_2(p) = \int_0^\infty \frac{ds}{2\pi}\, \frac{\rho(s)}{p^2 + s - i\epsilon}\,. \tag{11.15}$$

To that end, it is most convenient to consider the imaginary part originating from the $i\epsilon$ prescription

$$\frac{1}{x - i\epsilon} = \frac{1}{x} + i\pi\delta(x). \tag{11.16}$$

We thus find that the imaginary part of M_2 is directly related to the spectral function up to a factor of 2

$$\rho(-p^2) = 2\,\mathrm{Im}\, M_2(p). \tag{11.17}$$

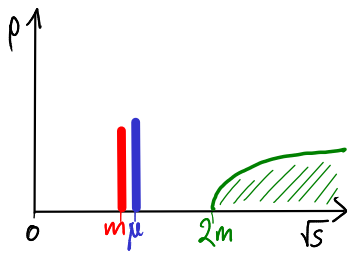
We can evaluate the imaginary part of M_2 using the derivative of the above identity for the $i\epsilon$ prescription

$$\frac{1}{(x - i\epsilon)^2} = \frac{1}{x^2} - i\pi\delta'(x). \tag{11.18}$$

We then find the spectral function

$$\begin{aligned}
\rho(s) &= 2\pi\delta(-s + \mu^2) - 2\pi\mu^2\kappa^2\,\mathrm{Re}\, I(s)\delta'(-s + \mu^2) \\
&\quad + \frac{2\mu^2\kappa^2\,\mathrm{Im}\, I(s)}{(-s + \mu^2)^2} + \dots \\
&= 2\pi\delta\big(s - \mu^2 + \mu^2\kappa^2\,\mathrm{Re}\, I(\mu^2)\big)\left(1 - \mu^2\kappa^2\ \mathrm{Re}\, I'(\mu^2)\right) \\
&\quad + \frac{2\mu^2\kappa^2\,\mathrm{Im}\, I(s)}{(s - \mu^2)^2} + \dots \\
&= 2\pi Z\delta\big(s - m^2\big) + \frac{2\mu^2\kappa^2\,\mathrm{Im}\, I(s)}{(s - \mu^2)^2} + \dots ,
\end{aligned} \tag{11.19}$$

which matches precisely with our expectations.



(11.20)

11.2 Loop Integral

We will now turn back to the loop integral $I(-p^2)$ and evaluate it.

Combining Denominators. A standard trick to proceed due to Feynman is to combine the denominators by virtue of a new integral. For two propagators we use the integral[3]

$$\frac{1}{AB} = \int_0^1 \frac{dz}{(zA + \bar{z}B)^2}\,, \qquad \bar{z} := 1 - z. \tag{11.21}$$

An alternative is the trick used by Schwinger to convert each numerator to an exponent where they automatically sum up

$$\frac{1}{A} = \int_0^\infty dz\, e^{-Az}\,. \tag{11.22}$$

The loop integral then takes the form with a single squared denominator[4]

$$I(-p^2) = \int_0^1 dz \int \frac{-id^4\ell}{32\pi^4} \frac{1}{\left(z\ell^2 + \bar{z}(p-\ell)^2 + \mu^2 - i\epsilon\right)^2}\,. \tag{11.23}$$

Now the denominator has quadratic (ℓ^2), linear ($\ell{\cdot}p$) and constant terms. We can remove the linear term and thus simplify the integral by a shift $\ell \to \ell + \bar{z}p$ of the integration variable[5]

$$I(-p^2) = \int_0^1 dz \int \frac{-id^4\ell}{32\pi^4} \frac{1}{\left(\ell^2 + z\bar{z}p^2 + \mu^2 - i\epsilon\right)^2}\,. \tag{11.24}$$

Momentum Integrals. Performing the momentum integrals is not very difficult. Let us start with the integral over the energy component. The integrand has double and single poles at

$$\ell_0 = \pm\sqrt{\vec{\ell}^{\,2} + z\bar{z}p^2 + \mu^2 - i\epsilon}\,. \tag{11.25}$$

The integration contour is along the real axis and passes right between the two poles. It decays sufficiently fast at $|l_0| \to \infty$ so that we can close the contour by a large semicircle in the upper or lower half of the complex plane. Either of the single poles contributes the same residue

$$I(-p^2) = \int_0^1 dz \int \frac{d^3\vec{\ell}}{64\pi^3} \frac{1}{\left(\vec{\ell}^{\,2} + z\bar{z}p^2 + \mu^2 - i\epsilon\right)^{3/2}}\,. \tag{11.26}$$

[3] There are similar formulas for more than two denominators and for higher powers.

[4] The advantage of this expression is that the new integrand is spherically symmetric for fixed value of z which simplifies integration drastically. The centre of the sphere, however, varies with z.

[5] For an infinite integration domain shifting the integration variable does not change the integral as long as it is convergent. For divergent integrals this point is subtle.

We notice that the integral merely reduces the exponent of the denominator by 1/2 and multiplies by a suitable overall factor. The remaining three spatial momentum integrals yield a very similar result, each of them reducing the exponent by 1/2. The last integral in fact is logarithmically divergent for large momenta. We have to cut it off at some bound $|\vec{l}| \simeq \Lambda_{\rm cut}$, and we obtain

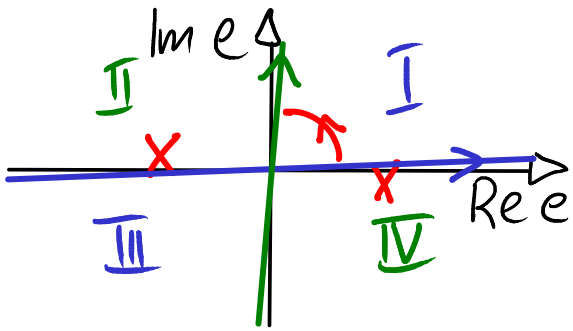
$$I(-p^2) = -\frac{1}{32\pi^2}\int_0^1 dz \log\frac{z\bar{z}p^2+\mu^2-i\epsilon}{\Lambda_{\rm cut}^2}\,. \tag{11.27}$$

For a large UV cutoff $\Lambda_{\rm cut}$ the integral diverges logarithmically.

Wick Rotation. Another trick which is commonly used is to rotate the integration contour for the energy ℓ_0 from the real axis to the imaginary axis

$$\ell_0 = i\ell_{\rm E,4}. \tag{11.28}$$

This Wick rotation is permissible since physical integrands are typically perfectly analytic in the first and third quadrants of the complex plane. For instance, poles of Feynman propagators are located slightly below the positive real axis in quadrant four or slightly above the negative real axis in quadrant two.



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By means of the residue theorem, the value of the integral does not change by the Wick rotation

$$\int d^4\ell\, F(\ell_0,\vec{\ell}) = \int i d^4\ell_{\rm E}\, F(i\ell_{\rm E,4},\vec{\ell}). \tag{11.30}$$

Applying this rotation to our loop integral we obtain an integral over 4-dimensional Euclidean space

$$I(-p^2) = \int_0^1 dz \int \frac{d^4\ell_{\rm E}}{32\pi^4}\,\frac{1}{\left(\ell_{\rm E}^2+z\bar{z}p^2+\mu^2-i\epsilon\right)^2}\,. \tag{11.31}$$

Now the integrand depends only on $|\ell_{\rm E}|$, and we can use rotational symmetry to replace the integral over three spherical angles at fixed $|\ell_{\rm E}|$ by the volume of a three-sphere $2\pi^2|\ell_{\rm E}|^3$

$$\int d^4\ell_{\rm E} F(|\ell_{\rm E}|) = 2\pi^2\int_0^\infty \ell_{\rm E}^3\, d\ell_{\rm E} F(\ell_{\rm E}). \tag{11.32}$$

For our integral this implies

$$I(-p^2) = \int_0^1 dz \int_0^\infty \frac{d\ell_{\rm E}}{16\pi^2} \frac{\ell_{\rm E}^3}{\left(\ell_{\rm E}^2 + z\bar{z}p^2 + \mu^2 - i\epsilon\right)^2} \,. \tag{11.33}$$

The integral over $\ell_{\rm E}$ is divergent again an needs to be cut off at $|\ell_{\rm E}| \simeq \Lambda_{\rm cut}$. The result is compatible with the above expression up to some minor adjustment of the cutoff parameter $\Lambda_{\rm cut}$.

Final Integral. Gladly the remaining integral over the Feynman parameter z can be performed for our simple integral yielding

$$\begin{aligned} I(-p^2) = &- \frac{1}{16\pi^2} \sqrt{\frac{p^2 + 4\mu^2 - i\epsilon}{-p^2}} \arctan \sqrt{\frac{-p^2}{p^2 + 4\mu^2 - i\epsilon}} \\ &- \frac{1}{32\pi^2} \log \frac{\mu^2}{\Lambda_{\rm cut}^2 e^2} \,. \end{aligned} \tag{11.34}$$

The integral is manifestly real for $0 < -p^2 < 4\mu^2$. It is also real for $-p^2 < 0$. However, for $-p^2 > 4\mu^2$ it develops an imaginary part

$$\operatorname{Im} I(-p^2) = \frac{1}{32\pi} \sqrt{\frac{-p^2 - 4\mu^2}{-p^2}} \,. \tag{11.35}$$

It signals the opening of the two-particle creation channel at $-p^2 > 4\mu^2$.

Spectral Function. We can now write the spectral function for our model at next-to-leading order

$$\rho(s) = 2\pi Z \delta\big(s - m^2\big) + \frac{\mu^2\kappa^2\theta(s - 4\mu^2)}{16\pi(s-\mu^2)^2} \sqrt{\frac{s - 4\mu^2}{s}} + \dots . \tag{11.36}$$

The two terms correspond to the asymptotic particle and the two-particle continuum. Let us consider its normalisation, we find

$$\int \frac{ds}{2\pi} \rho(s) = Z + \kappa^2 \frac{2\sqrt{3}\pi - 9}{288\pi^2} + \dots \tag{11.37}$$

Since $2\sqrt{3}\pi > 9$ we see that the correction term due to the two-particle continuum is indeed small and positive. Reassuringly the field strength renormalisation precisely compensates for the correction term

$$Z = 1 - \mu^2\kappa^2 I'(\mu^2) + \dots , \qquad I'(\mu^2) = \frac{2\sqrt{3}\pi - 9}{288\pi^2\mu^2} \,. \tag{11.38}$$

11.3 Regularisation and Renormalisation

Above, we have encountered a divergent integral $I(-p^2)$ and in order to evaluate it anyway, we somewhat arbitrarily introduced a momentum cutoff Λ_{cut}. Gladly the cutoff has only a mild impact on the function $I(-p^2)$

$$\frac{d}{d\Lambda_{\text{cut}}} I(-p^2) = \frac{1}{16\pi^2 \Lambda_{\text{cut}}} \,. \tag{11.39}$$

In particular, the dependence on Λ_{cut} does not mix at all with the dependence on the momenta and masses! This allowed us to extract some information from $I(-p^2)$ without caring too much about the cutoff.

Regularisation Schemes. In order to extract precise information from a QFT model at higher orders, one has to introduce a consistent regularisation scheme. Such a scheme should make all relevant quantities finite.

There are several schemes, e.g.:

- *Cutoff.* Our choice was to cut off momentum integrals at very large momenta (UV). Similarly one could cut off very small momenta (IR). Unfortunately, a cutoff is not easy to formulate consistently for all quantities. It is often used to quickly derive individual leading order results.
- *Pauli–Villars.* Replace Feynman propagators by a difference of two propagators

  $$\frac{1}{p^2 + m^2 - i\epsilon} \to \frac{1}{p^2 + m^2 - i\epsilon} - \frac{1}{p^2 + M^2 - i\epsilon} \,. \tag{11.40}$$

  For large M^2 and small p^2 the second propagator is suppressed. For large p^2, however, the two propagators almost cancel. Therefore this scheme suppresses UV divergences. It is similar to a UV cutoff, but it can be applied universally to Feynman diagrams.
- *Point Splitting.* In position space, the problem of UV divergences is related to putting several fields at the same point in spacetime. By separating the field insertion points in the action by a tiny amount, UV divergences can be avoided.
- *Lattice.* For the lattice regulator one approximates infinite spacetime by a finite lattice. For finitely many degrees of freedom there cannot be divergences, neither from the UV (finite spacing), nor from the IR (finite extent).
- *Dimensional Regularisation.* The types and degrees of divergences depend crucially on the number of spacetime dimensions D. In the dimensional regularisation scheme, one works in a spacetime of dimension D, where D is taken to be an unconstrained real number. Observables become functions of D, and divergences appear as poles in the D-dependence, e.g. $1/(D-4)$. Although the definition of this scheme is somewhat abstract, it is one of the favourite ones because it works well in almost all circumstances.
- *Finite Observables.* Sometimes divergences can be avoided by considering physical observables only. In our example, one could try to argue that all observables can be deduced from $I'(s)$ which is perfectly finite. The constant term of $I(s)$ is an integration constant of $\int ds\, I'(s)$. The divergence happens to be located precisely in this undetermined coefficient.

In a regularised QFT, all observables are perfectly finite. However, they are not quite what we are interested in, we are interested in observables of the original QFT.

Renormalisation. The next step called renormalisation is to somehow absorb the divergences consistently. To that end the most important insights are the following:

- All physically relevant information and all observables for a QFT model are encoded into its quantum correlation functions.
  Example. The spectrum of asymptotic particles is encoded into time-ordered two-point functions. Moreover, the scattering matrix can be derived from the poles of higher-point functions.
- The Lagrangian and the action are devices to derive suitable correlation functions. They are not fundamental objects, in particular their parameters such as masses and coupling constants are not directly observable.[6]
  Example. The mass terms (μ) in the Lagrangian do not exactly reflect the masses (m) of asymptotic particles.
- Correlation functions depend on the so-called bare parameters of the Lagrangian. One should tune the parameter values such that the correlation functions behave as expected. At the end of the day, the numerical values of the parameters are not important.
  Example. One would adjust μ and κ such that the physical mass m has the desired physical value.
- In terms of differential geometry: The parameters of a QFT form a manifold. A Lagrangian description (with a particular regularisation, renormalisation and, where applicable, a particular gauge fixing scheme) is a chart of the manifold. The parameter values correspond to coordinates on this particular chart. There is, however, no universal meaning to coordinates without reference to the specific chart.

In this picture, renormalisation is the step to adjust the Lagrangian parameters to the physical parameters. In the regularised and finite QFT this step is well-defined. We express the bare Lagrangian parameters in terms of the physical parameters, e.g.

$$\mu = \mu(m, \Lambda_{\text{cut}}) = m + \tfrac{1}{2} m\kappa^2 I(m^2) + \dots . \tag{11.41}$$

with

$$I(m^2) = \frac{2 - \pi/\sqrt{3}}{32\pi^2} - \frac{1}{32\pi^2} \log \frac{m^2}{\Lambda_{\text{cut}}^2} . \tag{11.42}$$

Running Coupling. We can now remove the regulator by sending the regularisation parameters to some appropriately chosen limit. We shall keep the physical parameters fixed in the limit, but the resulting bare parameters may well

[6] Some traces of the complete Lagrangian or action may remain valid in the QFT, such as the exact equations of motion and normalisation of the interacting field, as well as Noether's theorem.

be divergent

$$\mu(m, \Lambda_{\text{cut}}) \to \infty \quad \text{as} \quad \Lambda_{\text{cut}} \to \infty. \tag{11.43}$$

This by itself is not a problem, since we attribute no meaning to μ. We just need to keep in mind that the definition of bare parameters such as μ depends on a scale such as Λ_{cut}. Changing the scale must be compensated by a change in the bare parameter. This effect is called running of a coupling constant. In our case the running is governed by the equation

$$\frac{d\mu}{d\Lambda_{\text{cut}}} = \frac{m\kappa^2}{16\pi^2\Lambda_{\text{cut}}} + \dots. \tag{11.44}$$

Such an equation is often written in logarithmic form

$$\frac{d\log\mu}{d\log\Lambda_{\text{cut}}} = \frac{\kappa^2}{16\pi^2} + \dots \tag{11.45}$$

which suggests the scaling behaviour

$$\mu \sim m(\Lambda_{\text{cut}}/m)^{\kappa^2/16\pi^2+\dots}. \tag{11.46}$$

Note that the dependence of the bare mass μ on the physical mass m has become non-linear by quantum effects. In the quantum field theory the mass term has acquired a so-called anomalous dimension.

A similar effect can be observed for coupling constants governing the interactions of several particles. In quantum field theory one may find an anomalous dependence on the particle momenta, e.g. $\lambda \sim (p/\Lambda)^{*\kappa^2+\dots}$. It means that one will measure a different effective coupling strength depending on which length or energy scale the interaction is probed (e.g. the energy of the probe photon). Naturally, one would like to define a universally valid coupling strength to appear in the Lagrangian, e.g. by considering the limit of very high or very low energies. However, in this limit, the coupling strength often diverges. Therefore one needs to define the coupling constant at a particular energy scale Λ, and the value of this coupling constant depends on Λ. This effect is called a running coupling constant.

Renormalisability. The question is whether all physical quantities remain finite in the limit $\Lambda_{\text{cut}} \to \infty$, and whether they are independent of the chosen regularisation scheme. We can only adjust one bare parameter per physical parameter, are there sufficiently many bare parameters to absorb all the divergences?

- In the case of our model the answer is yes.
- QFT models where all divergences can be absorbed are called renormalisable.
- In principle, one can introduce further terms and couplings in the Lagrangian to compensate for more and more divergences. As long as only finitely many terms are needed to absorb the divergences at all perturbative orders, the model is called renormalisable.

- Some models, such as General Relativity, appear to require infinitely many coupling constants to absorb all divergences. These models are called non-renormalisable. Here one would need infinitely many measurements to adjust infinitely many parameters, and effectively the model loses its predictive power.

Before discussing which models are renormalisable, let us consider some technical aspects of absorbing divergences into the coupling constants.

Localised Divergences. In principle, we know how to absorb divergences by writing the bare parameters of the Lagrangian as functions of the physical parameters. Let us discuss the origin of the divergence in detail.

To that end we investigate the integral $I(-p^2)$ more closely. The overall dependence on p^2 is some inverse trigonometric function.[7] However, the divergent or cutoff-dependent contribution to $I(-p^2)$ is much simpler

$$\frac{d}{d\Lambda_{\text{cut}}} I(-p^2) = \frac{1}{16\pi^2 \Lambda_{\text{cut}}} \,. \tag{11.47}$$

It is actually independent of p^2!

This behaviour is in fact general; divergences typically couple to polynomials of the momenta only. This statement becomes more meaningful when translated to position space: Any polynomial of the momenta translates to a localised distribution such as $\delta^4(x)$ under a Fourier transformation.

In terms of the loop integral in position space we can localise the origin of the divergence. The loop integrand diverges as r^{-4} when the two vertices are nearby at a distance of r, otherwise it is perfectly finite.

(11.48)

This divergence cannot be compensated by the measure $d^4x \sim r^3 dr$, and the region $r \approx 0$ contributes an infinite amount to the integral. The divergence is therefore localised in spacetime, and it can be absorbed by a suitable local term in the Lagrangian.

(11.49)

Counterterms. Indeed we see that we can absorb the divergence by a suitable definition of the bare mass term $\mu = \mu(m, \Lambda_{\text{cut}})$ in the Lagrangian. The prescription may be somewhat confusing because the integral $I(-p^2)$ depends on the mass μ, so the definition of μ apparently is implicit. For our calculations we use the interaction picture where we decided to split the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ into a free and an interaction contribution

$$\mathcal{L}_0 = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2, \qquad \mathcal{L}_{\text{int}} = -\tfrac{1}{6}\kappa\mu\phi^3 - \tfrac{1}{24}\kappa^2\lambda\phi^4. \tag{11.50}$$

[7] We were extremely lucky to find a simple function. Typically one finds much more complicated special functions such as polylogarithms or hypergeometric functions. And this only if one is lucky. Often the encountered integrals lead to functions which do not even have a name.

However, we are not forced to do this naive split, we are free to choose any free field as a reference. In this sense, it makes perfect sense to choose the asymptotic field with the physical mass m as a reference

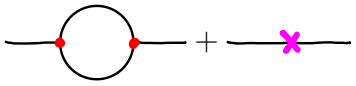
$$\mathcal{L}_{\text{as}} = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2. \tag{11.51}$$

This will automatically position all the poles due to Feynman propagators at the desired physical location. Now we have to add a compensating mass term to the interaction terms[8]

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}\mu' m^2\phi^2 - \tfrac{1}{6}\kappa m\phi^3 - \tfrac{1}{24}\lambda\kappa^2\phi^4. \tag{11.52}$$

The new term is called a counterterm. Its role is to compensate all potential mass shifts due to loop effects. The mass of the reference field will thus conveniently be the physical mass to all orders.[9] In this picture, the loop integral $I(-p^2)$ is defined directly in terms of the physical mass m instead of μ.

When including the counterterm in our example, we obtain the following one-loop contributions[10]



(11.53)

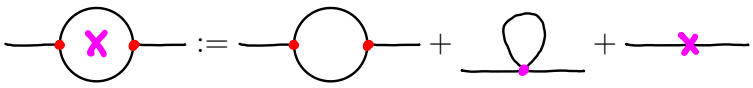
and the following corrected two-point function

$$M_2(p) = \frac{1}{p^2 + m^2 + \mu' m^2 - m^2\kappa^2 I(-p^2) - i\epsilon} + \dots. \tag{11.54}$$

We impose the consistency equation that the physical mass equals the asymptotical mass

$$m^2 = m^2 + \mu' m^2 - m^2\kappa^2 I(m^2) + \dots, \tag{11.55}$$

which is solved by $\mu' = \kappa^2 I(m^2)$. Concerning the counterterm, the rule is that whenever a bubble with two legs appears in a Feynman graph, there is a compensating counterterm.[11]



(11.56)

It therefore makes sense to introduce a subtracted bubble integral

$$I_{\text{sub}}(-p^2) := I(-p^2) - I(m^2). \tag{11.57}$$

[8]The definition of μ', κ and λ has changed, but this does not matter since the values of the bare parameters are not directly measurable.

[9]Only the mass is stabilised by the counterterm. The two Feynman diagrams do not cancel exactly because the loop integral has a complicated dependence on p^2 whereas the counterterm is a constant function.

[10]The loop order does not necessarily refer to the literal number of loops in a Feynman graph. It makes sense to also count counterterms to loop divergences as loops. The loop order commonly refers to the order in a small coupling constant, in our case κ^2.

[11]We have reintroduced the contribution from the tadpole graph which is equivalent to a mass term with infinite mass and can therefore be absorbed entirely into a suitable redefinition of μ'.

Obviously this function is finite and satisfies

$$I_{\text{sub}}(m^2) = 0. \tag{11.58}$$

The corrected two-point function then takes the simplified form

$$M_2(p) = \frac{1}{p^2 + m^2 - m^2\kappa^2 I_{\text{sub}}(-p^2) - i\epsilon} + \dots \tag{11.59}$$

We can even go one step further and decide to add a counterterm for the kinetic term

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}\zeta(\partial\phi)^2 - \tfrac{1}{2}\mu' m^2\phi^2 - \tfrac{1}{6}\kappa m\phi^3 - \tfrac{1}{24}\lambda\kappa^2\phi^4. \tag{11.60}$$

This term changes the overall normalisation of the field $\phi(x)$, and allows us to normalise the residue of the asymptotic particle pole in the two-point function to 1 at all orders. In this case, we can drop the field strength renormalisation factor Z by setting it to 1.

In conclusion, the asymptotic Hamiltonian describes the canonically normalised fields with appropriate physical masses. The interaction Hamiltonian contains all types of allowable interactions terms. Their parameters are tuned to stabilise the masses and normalisations and to match with physical interaction processes.

Power Counting. Can we understand under which circumstances a QFT model is renormalisable? We can use a crude argument in terms of the mass dimensions of interaction terms.

A Feynman diagram evaluates to an integral over a rational function of the momenta and masses of the particles.

- Earlier, we have discussed that UV divergences of the integral are polynomials of the momenta.
- Furthermore, we can argue that particle masses in the denominators can be safely ignored for the purposes of UV divergences.
- Evidently, the coupling constants will appear as overall prefactors.

Altogether this implies that the structure of UV divergences of a loop integral is given by polynomials in the momenta and the masses

$$\frac{dI}{d\log\Lambda} \in \text{Poly}(\alpha_k, p_k, m_k). \tag{11.61}$$

The polynomials in the momenta p_k determine the appropriate local counterterm.

The class of potential divergences is restricted by the following consideration:

- The mass dimension of the integral is determined by the Feynman diagram. It is a fixed number usually bounded from above by the number of spacetime dimensions D, in our case $D = 4$.
- Suppose that all terms of the Lagrangian have mass dimension bounded from above by D. Equivalently, all coupling constants α_k have a non-negative mass dimension.

- Then the overall polynomial divergence terms must have non-negative mass dimension. The remaining mass dimension must be carried by the momenta and masses.
- This implies that there is only a very restricted set of momentum polynomials which can carry divergences. In other words, only few counterterms are needed to compensate the divergences.
- These counterterms have mass dimension bounded from above by D. The counterterms couplings in turn have non-negative mass dimension.
- Usually, there are finitely many such terms, and therefore such models are renormalisable.

Note that there is a crucial difference between coupling constants with positive mass dimension and dimensionless coupling constants. As the dimension of coupling constants accumulates, dimensionful coupling constants contribute divergences only for a specific range of low perturbative orders. When there are also dimensionless coupling constants, counterterms are required for arbitrary loop orders.

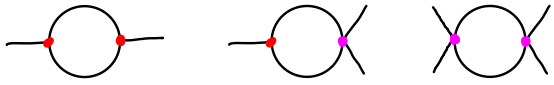
In our example, we have included all interaction terms of mass dimension bounded from above by 4. For each divergence which can possibly arise, there is a corresponding counterterm.

When considering Feynman diagrams which contain a divergent loop along with some other structures, it can be shown that the loop can be replaced by a universal counterterm corresponding to the loop to cancel the divergence. A subtle issue at higher loops are overlapping loops, where it may not be evident which counterterms to use. Gladly, it can be shown that this situation does not leave behind divergences which cannot be accounted for.

11.4 Vertex Renormalisation

Let us briefly discuss how to renormalise the remaining divergences at the one-loop level in our example.

Divergent Interactions. By power counting arguments we can derive that divergences can only appear for two-sided loops.[12] Loops with three or more legs are perfectly finite. There are only three potentially divergent terms in our model.

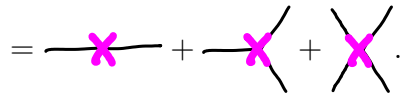


(11.62)

One of them we have already discussed, the other two integrals can be made finite by adding appropriate counterterms of the form ϕ^3 and ϕ^4 to the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{int}} &= -\tfrac{1}{6}\kappa m\phi^3 - \tfrac{1}{24}\lambda\kappa^2\phi^4 + \mathcal{L}_{\text{ct}},\\ \mathcal{L}_{\text{ct}} &= -\tfrac{1}{2}\mu' m^2\phi^2 - \tfrac{1}{6}\kappa' m\phi^3 - \tfrac{1}{24}\lambda'\kappa^2\phi^4\end{aligned}$$

[12] One-sided tadpoles can be removed entirely by suitable counterterms.



(11.63)

Suitable counterterm coefficients to make all observables finite at one loop read

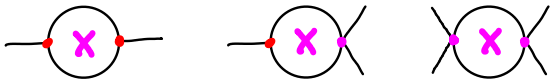
$$\mu' = \kappa^2 I(m^2), \qquad \kappa' = 3\kappa^3\lambda I(m^2), \qquad \lambda' = 3\kappa^2\lambda^2 I(m^2). \tag{11.64}$$

As discussed above, μ' is determined by a stable physical mass at m. There is no similar universal condition for the coefficients κ' and λ'; any finite shift w.r.t. the above values is permissible, it merely leads to a reparametrisation of our model.

We have already seen that the ϕ^2 counterterm effectively replaces the loop integral by a finite subtracted loop integral

$$I(-p^2) \to I_{\text{sub}}(-p^2). \tag{11.65}$$

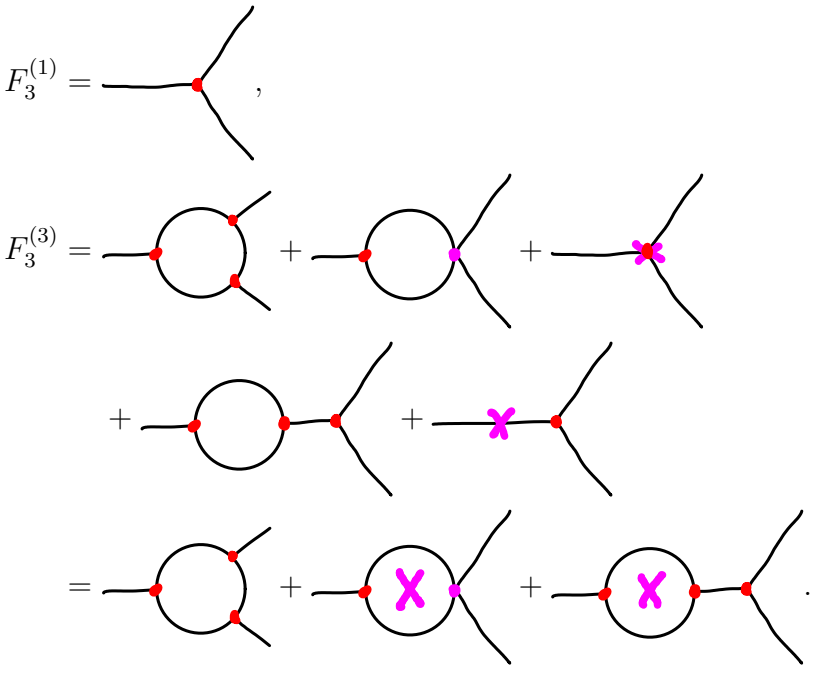
Exactly the same replacement is achieved by our above choice of counterterms for ϕ^3 and ϕ^4.



(11.66)

Three-Point Function. Let us briefly consider the resulting one-loop contributions to the three-point function.

$$F_3 = F_3^{(1)} + F_3^{(3)} + \ldots,$$



(11.67)

The latter two terms involve bubbles only for which we know the integral already. The counterterms make both integrals finite.[13] The first triangle integral

[13]Note that because $I_{\text{sub}}(m^2) = 0$ the internal Feynman propagator which connects the bubble to the 3-vertex is precisely cancelled, and only one Feynman propagator with associated asymptotic particle pole remains for each leg.

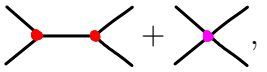
$\sim \int d^4\ell/\ell^6$ is UV finite by itself. In fact it is the most complicated contribution

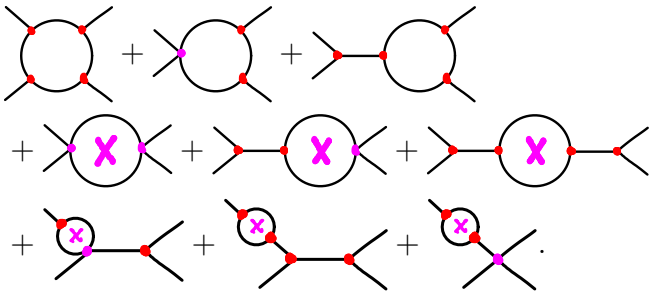
$$\begin{aligned} F_3^{(3)} = & -i\kappa^3 m^3 (2\pi)^4 \delta^4(p_1+p_2+p_3) \\ & \cdot \int \frac{-id^4\ell}{(2\pi)^4} \frac{1}{\ell^2+m^2-i\epsilon} \\ & \cdot \frac{1}{(\ell+p_1)^2+m^2-i\epsilon} \frac{1}{(\ell-p_3)^2+m^2-i\epsilon} \end{aligned} \tag{11.68}$$

and the only one which allows all three momenta to interact non-trivially.

Four-Point Function. There are many diagrams contributing to a four-point process. Here we can merely plot all the graphs (up to permutations of the external legs).

$F_4 = F_4^{(2)} + F_4^{(4)} + \ldots,$

$F_4^{(2)} =$  ,

$F_4^{(4)} =$  .

(11.69)