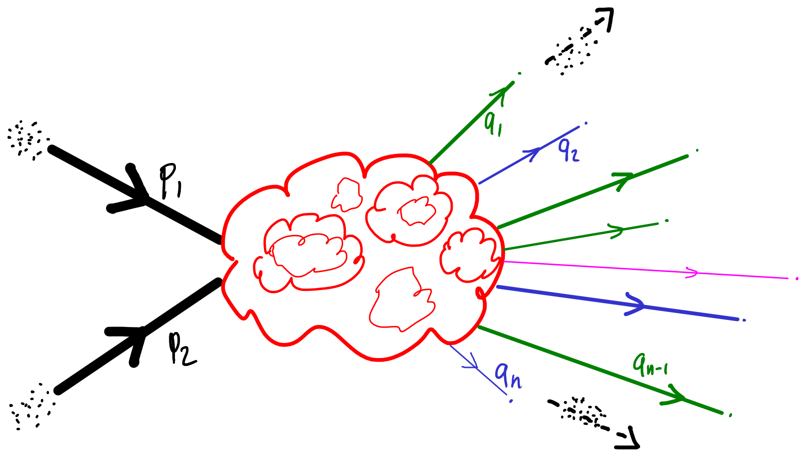
9 Particle Scattering

A goal of this course is to understand how to compute scattering processes in particle physics.

9.1 Scattering Basics

Setup. The usual setup for scattering experiments at particle colliders is the following:



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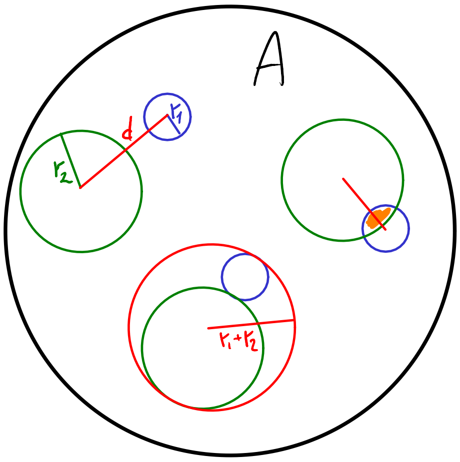
- Two bunches of particles are accelerated to high or relativistic velocities and made to collide.
- Whenever two particles from the bunches come very close, they produce some complicated interacting quantum state.
- After a while this state evolves into several particles moving away in various directions.
- The outgoing particles of each scattering event are measured and recorded.

Some additional comments:

- Quantum mechanics is probabilistic, so a large number of particle collisions must be measured.
- To measure collisions of three or more particles would be technically challenging because they would all have to be focused within a tiny region of space simultaneously. The likelihood for two particles to scatter is much higher.
- By Lorentz symmetry, the directions of the two ingoing momenta $\vec{p}_{1,2}$ can be adjusted arbitrarily. In some reference frame, the momenta will be parallel and along the z-axis. The relevant quantity is the centre of mass energy squared $s = -(p_1 + p_2)^2$. The highest energies $\sqrt{s}$ are obtained where the collisions are head-on with equal but opposite momenta. For practical purposes, the particles can have momenta of different magnitude or one of the two bunches could be a fixed target at rest.

- The particle momenta in the beam are not perfectly aligned. By the uncertainty principle this is actually impossible if the beam is also focused on a finite area.
- The particle detectors are not perfect: They have a certain spatial and temporal resolution. They measure the energy and momenta at a certain resolution. They may not be able to detect and distinguish all kinds of particles; they may miss some particles; they may misidentify some. Scattered particles along the beam direction are hardest to detect.

Cross Sections. How to quantify scattering? Consider a simple classical scattering experiment:



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- Take two hard balls of radii r_1, r_2.
- Throw them towards each other along the z axis in opposing directions.
- Depending on the transverse offset d, the balls will either hit ($d < r_1 + r_2$) or miss ($d > r_1 + r_2$).[1]
- When the balls hit, they bounce off in different directions.

Quantum mechanics is probabilistic, there cannot be such deterministic output. One has to repeat the experiment many times or perform an experiment with many identical particles and count:

- Accelerate two bunches of n_1 and n_2 particles.
- Focus each bunch on a cross-sectional area of A.
- Repeat the experiment n_{ex} times.
- Count the number of individual scattering events N.

The expectation value for N is

$$N = \frac{n_{\text{ex}} n_1 n_2 \sigma}{A}, \tag{9.3}$$

where the characteristic quantity is the scattering cross section σ. For two classical hard balls one obtains $\sigma = \pi(r_1 + r_2)^2$: Given the transverse position of the first ball, the second ball must be within an area of σ to make the two collide.

In collider experiments one measures scattering cross sections

[1]More accurately, at a near miss, the flow of air will also deform the balls' trajectories slightly.

- Total or inclusive cross sections σ simply count the number of overall collision events.
- Differential cross sections $d\sigma$ measure the number of events where the outgoing particles have predetermined momenta.[2] The definition of $d\sigma$ depends on the number of outgoing particles. The so-called phase space is furthermore constrained by Poincaré symmetry.
- One may even resolve the polarisation of the outgoing particles and measure polarised cross sections.

9.2 Cross Sections and Matrix Elements

The computation of the scattering cross section is not straight-forward. Naively, we prepare initial and final states with definite momenta p_1, p_2 and $q_1, \ldots, q_n$ at some times t_{in} and t_{out} in the distant past and distant future

$$\langle \mathrm{f}| \sim \langle q_1, \ldots, q_n|, \qquad |\mathrm{i}\rangle \sim |p_1, p_2\rangle. \tag{9.4}$$

The probability is given by the square of the correlator $\langle \mathrm{f}| \exp(-iH(t_{\text{out}} - t_{\text{in}}))|\mathrm{i}\rangle$

$$\sigma \sim |\langle \mathrm{f}| \exp(-iH(t_{\text{out}} - t_{\text{in}}))|\mathrm{i}\rangle|^2 \tag{9.5}$$

For initial and final states with definite momenta, the correlator contains a delta function $\delta^4(P_{\text{in}} - P_{\text{out}})$ to conserve momentum. It cannot be squared because this would result in a factor of $\delta^4(0) = \infty$. We know that such factors represent some volume of spacetime relevant to the problem. A proper treatment requires the use of wave packets. They actually account for the finite extent of the ingoing bunches, namely the cross-sectional area A, and for the finite resolution of the detector. The factor $\delta^4(0)$ represents this area A among others.

A somewhat tedious calculation in terms of wave packets yields a meaningful result for the differential cross section of $2 \to n$ scattering. At the end of the day, the wave packets can be focused to definite momenta[3]

$$d\sigma = \frac{(2\pi)^4 \delta^4(P_{\text{in}} - P_{\text{out}})}{4|e(\vec{p}_1)\vec{p}_2 - e(\vec{p}_2)\vec{p}_1|} \prod_{k=1}^{n} \frac{d^3\vec{q}_k}{(2\pi)^3\, 2e(\vec{q}_k)} |M|^2. \tag{9.6}$$

Here M is the appropriate element of the scattering matrix with the momentum-conserving delta function stripped off

$$\lim_{t_{\text{in,out}} \to \mp\infty} \langle \mathrm{f}| \exp(-iH(t_{\text{out}} - t_{\text{in}}))|\mathrm{i}\rangle = (2\pi)^4 \delta^4(P_{\text{in}} - P_{\text{out}}) iM. \tag{9.7}$$

The normalisation is such that in the free theory the correlator for $n = 2$ two final state particles equals[4]

$$2e(\vec{p}_1)\, 2e(\vec{p}_2)\, (2\pi)^6 \delta^3(\vec{p}_1 - \vec{q}_1)\delta^3(\vec{p}_2 - \vec{q}_2). \tag{9.8}$$

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[2] The direction of the scattered classical balls is determined by the impact parameter d, and hence certain regions of the scattering cross section correspond to specific angles. In quantum mechanics this is mostly a matter of probability.

[3] This quantity is not invariant under Lorentz transformations due to the denominator $|e(\vec{p}_1)\vec{p}_2 - e(\vec{p}_2)\vec{p}_1|$. Nevertheless it transforms like an area as it should.

[4] This contribution representing no scattering is actually removed from M for $2 \to 2$ particle scattering.

The formula simplifies for $2 \to 2$ scattering in the centre of mass frame

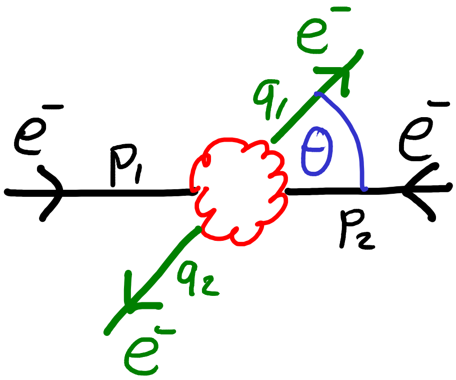
$$\frac{d\sigma}{d\Omega} = \frac{1}{4|e(\vec{p}_1)\vec{p}_2 - e(\vec{p}_2)\vec{p}_1|} \frac{|\vec{q}_1|}{16\pi^2\sqrt{s}} |M|^2. \tag{9.9}$$

Here $d\Omega$ represents the spherical angle element of the direction of outgoing particle 1, and $s = -P_{\text{in}}^2 = -(p_1 + p_2)^2$ is the centre of mass energy squared. It becomes even simpler in case all four particles are identical

$$\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2 s}\,. \tag{9.10}$$

9.3 Electron Scattering

We can now compute some realistic scattering event in Quantum Electrodynamics. We shall consider scattering of two electrons into two electrons (Møller scattering).[5]



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Here, we will not distinguish the two polarisation modes of the electron spin. One might as well consider the polarised cross section, but the experimental setup as well as the theoretical calculation is more challenging.

Initial and Final States. To prepare the initial and final states we use the interaction picture. The free reference field provides the creation and annihilation operators for the in- and outgoing particles which do not interact when sufficiently far away. Moreover the initial and final states will be practically independent of t_{in} and t_{out} as long as the latter are sufficiently large.

The initial state is composed from two ingoing electrons

$$|\text{i}\rangle = a_\alpha^\dagger(\vec{p}_1) a_\beta^\dagger(\vec{p}_2)|0\rangle. \tag{9.12}$$

The electrons have definite momenta p_1, p_2. Let us assume they are in their centre of mass frame with momenta aligned along the z axis

$$p_{1,2} = (e, 0, 0, \pm p) \tag{9.13}$$

[5]Depending on conventions, our calculation may also represent positron-positron scattering. Obviously, the cross section is exactly the same by charge conjugation symmetry.

where $e^2 = p^2 + m^2$. The polarisations α, β are required to set up the state properly. We will not care about them, so we should eventually *average* over all ingoing polarisation configurations.

We want to probe the final state for two outgoing electrons

$$\langle \mathrm{f}| = \langle 0| a_\delta(\vec{q}_2) a_\gamma(\vec{q}_1). \tag{9.14}$$

In the centre of mass frame they will escape in two opposite directions with the same magnitude p of momentum. Due to rotational symmetry around the z axis,[6] we only need to probe for particles in the x-z plane

$$q_{1,2} = (e, \pm p \sin\theta, 0, \pm p \cos\theta). \tag{9.15}$$

Again we shall not care about the polarisations. We therefore have to *sum* over all outgoing polarisation configurations.

For a fixed particle momentum p or energy e, we will be interested in the angular distribution of outgoing particles. Due to rotational symmetry the differential cross section $d\sigma/d\Omega$ must be an even function of the scattering angle θ alone. This function also has the symmetry $\theta \to \pi - \theta$ because the outgoing particles are indistinguishable

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}(\theta) = \frac{d\sigma}{d\Omega}(-\theta) = \frac{d\sigma}{d\Omega}(\pi - \theta). \tag{9.16}$$

Time Evolution. We now insert the time evolution operator U_{int} of the interaction picture between the initial and final states to determine the probability amplitude[7]

$$F = \langle \mathrm{f}| U_{\mathrm{int}}(t_{\mathrm{out}}, t_{\mathrm{in}}) |\mathrm{i}\rangle = (2\pi)^4 \delta^4(P_{\mathrm{in}} - P_{\mathrm{out}}) iM. \tag{9.17}$$

The matrix element is a function of the momenta and the polarisations $M_{\alpha\beta\gamma\delta}(p_1, p_2, q_1, q_2)$.[8]

The expansion of the amplitude at leading order reads simply

$$\begin{aligned} F^{(0)} &= \langle \mathrm{f}|\mathrm{i}\rangle \\ &= 2e(\vec{p}_1)\, 2e(\vec{p}_2)\, (2\pi)^6 \delta^3(\vec{p}_1 - \vec{q}_1) \delta^3(\vec{p}_2 - \vec{q}_2) \\ &\quad - 2e(\vec{p}_1)\, 2e(\vec{p}_2)\, (2\pi)^6 \delta^3(\vec{p}_1 - \vec{q}_2) \delta^3(\vec{p}_2 - \vec{q}_1). \end{aligned} \tag{9.18}$$

The contribution from the free theory represents the situation when the two particles pass by each other without scattering at all. Note that there are two terms corresponding to the fact that the particles are indistinguishable.

At first perturbative order the matrix element vanishes

$$\begin{aligned} F^{(1)} &= i\langle \mathrm{f}| S_{\mathrm{int}} |\mathrm{i}\rangle = i \int d^4x\, \langle \mathrm{f}| \mathcal{L}_{\mathrm{int}}(x) |\mathrm{i}\rangle \\ &= iq \int d^4x\, \langle \mathrm{f}| A_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x) |\mathrm{i}\rangle = 0 \end{aligned} \tag{9.19}$$

[6] We will not measure polarisations which would otherwise break the symmetry.

[7] The conventional factor of i typically makes the leading contributions to M (mostly) real.

[8] We can write it as a function of all the external momenta noting that we shall only evaluate it for $p_1 + p_2 = q_1 + q_2$.

because there is a single electromagnetic field which cannot contract to anything else and thus directly annihilates either of the vacua.

Second Order. For the next order we insert two interaction Lagrangians

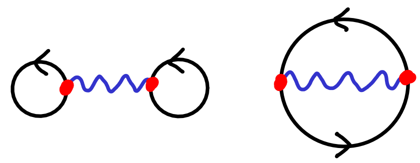
$$F^{(2)} = \tfrac{1}{2}i^2 \int d^4x\, d^4y\, \langle \mathrm{f}|\mathrm{T}\big(\mathcal{L}_{\mathrm{int}}(x)\mathcal{L}_{\mathrm{int}}(y)\big)|\mathrm{i}\rangle. \tag{9.20}$$

Each of the interaction Lagrangians contains an electromagnetic field. As they would otherwise annihilate the vacua, they have to be contracted via Wick's theorem (for the field A) by a Feynman propagator

$$\begin{aligned} &\tfrac{1}{2}i^2q^2 \int d^4x\, d^4y\, \langle \mathrm{f}|\mathrm{T}\big(A_\mu(x)\bar\psi(x)\gamma^\mu\psi(x)\, A_\nu(y)\bar\psi(y)\gamma^\nu\psi(y)\big)|\mathrm{i}\rangle \\ &= \tfrac{1}{2}iq^2 \int d^4x\, d^4y\, G^{\mathrm{F}}_{\mu\nu}(x-y)\langle \mathrm{f}|\mathrm{T}\big(\bar\psi(x)\gamma^\mu\psi(x)\, \bar\psi(y)\gamma^\nu\psi(y)\big)|\mathrm{i}\rangle. \end{aligned} \tag{9.21}$$

Next, Wick's theorem should be applied to the time-ordered spinor fields yielding several contributions:

- There are two vacuum bubble contributions.



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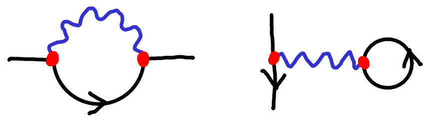
  These vacuum processes take place everywhere and all the time, and they do not interact with the scattering process. As discussed earlier, they must be discarded.
- There are two correction terms with two remaining external fields.



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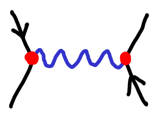
  They contribute to two point functions of spinor fields, but cannot be non-trivial functions of all the four scattering particle momenta. Here they contribute only to forward scattering, and we can safely ignore their contribution. We will discuss their relevance later.
- Finally, there is one connected diagram.



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  This is the leading non-trivial contribution to the scattering process.

Connected Contribution. In our case, the connected diagram is obtained by replacing time ordering by normal ordering

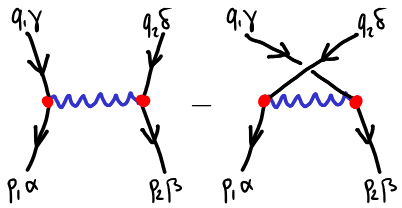
$$\tfrac{1}{2} i q^2 \int d^4x\, d^4y\, G^{\mathrm{F}}_{\mu\nu}(x-y) \langle \mathrm{f} | \mathrm{N}\big(\bar{\psi}(x)\gamma^\mu \psi(x)\, \bar{\psi}(y)\gamma^\nu \psi(y)\big) | \mathrm{i} \rangle. \tag{9.25}$$

Now we need to contract the fields with the external particles. This is achieved by the following two anti-commutators which follow from the mode expansion of the free Dirac field

$$\begin{aligned} \{\bar{\psi}(x), a^\dagger_\alpha(\vec{p})\} &= e^{-ip\cdot x} \bar{v}_\alpha(\vec{p}), \\ \{a_\alpha(\vec{q}), \psi(x)\} &= e^{iq\cdot x} v_\alpha(\vec{q}). \end{aligned} \tag{9.26}$$

Putting everything together we obtain the matrix element

$$\begin{aligned} F^{(2)}_{\mathrm{conn}} = i q^2 \int d^4x\, d^4y\, G^{\mathrm{F}}_{\mu\nu}(x-y) e^{iq_1\cdot x + iq_2 \cdot y - ip_1\cdot x - ip_2 \cdot y} \\ \cdot\, \bar{v}_\alpha(\vec{p}_1)\gamma^\mu v_\gamma(\vec{q}_1)\, \bar{v}_\beta(\vec{p}_2)\gamma^\nu v_\delta(\vec{q}_2) - \ldots \end{aligned}$$

$=$  (9.27)

The omitted term takes the same form, but with the two outgoing particles exchanged ($q_1 \leftrightarrow q_2$, $\gamma \leftrightarrow \delta$). The reason for the doubling of terms is that two identical types of particles are scattered. The two-particle wave function is anti-symmetric because the particles are fermionic.

The two remaining integrals are Fourier transforms. One of them transforms the Feynman propagator to momentum space. The other one generates the momentum conserving delta function. Altogether the integrals yields

$$(2\pi)^4 \delta^4(P_{\mathrm{out}} - P_{\mathrm{in}})\, G^{\mathrm{F}}_{\mu\nu}(q_1 - p_1). \tag{9.28}$$

We now separate off the momentum conserving delta function and write the matrix element M [9]

$$M_{\alpha\beta\gamma\delta} = \frac{q^2 \eta_{\mu\nu}}{(p_1 - q_1)^2}\, \bar{v}_\alpha(\vec{p}_1)\gamma^\mu v_\gamma(\vec{q}_1)\, \bar{v}_\beta(\vec{p}_2)\gamma^\nu v_\delta(\vec{q}_2) - \ldots . \tag{9.29}$$

We could try to evaluate the various spinor products. It turns out to be much simpler to square the matrix element first

$$|M|^2 = \frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} M_{\alpha\beta\gamma\delta} M^*_{\alpha\beta\gamma\delta}. \tag{9.30}$$

The factor of $1/2^2$ originates from averaging over the polarisations of the ingoing particles.

[9] The $i\epsilon$ prescription for the Feynman propagator will not be relevant here.

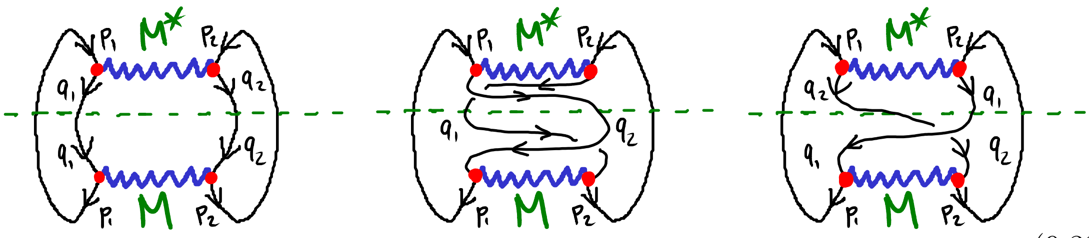
Sum over Polarisations. A pleasant feature of the polarisation sums is that they can be performed by the completeness relations for spinor solutions

$$\sum_{\alpha} v_\alpha(\vec{p})\bar{v}_\alpha(\vec{p}) = p\cdot\gamma - m. \tag{9.31}$$

We obtain the following three terms

$$|M|^2 = \frac{q^4 T_{tt}}{4(p_1-q_1)^4} + \frac{q^4 T_{uu}}{4(p_1-q_2)^4} - \frac{q^4 T_{tu}}{2(p_1-q_1)^2(p_1-q_2)^2}\,, \tag{9.32}$$

corresponding to the diagrams



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The spinor products have turned into the traces

$$\begin{aligned} T_{tt} = {} & \operatorname{tr}[(p_1\cdot\gamma - m)\gamma_\mu(q_1\cdot\gamma - m)\gamma_\nu] \\ & \cdot \operatorname{tr}[(p_2\cdot\gamma - m)\gamma^\mu(q_2\cdot\gamma - m)\gamma^\nu], \\ T_{uu} = {} & \operatorname{tr}[(p_1\cdot\gamma - m)\gamma_\mu(q_2\cdot\gamma - m)\gamma_\nu] \\ & \cdot \operatorname{tr}[(p_2\cdot\gamma - m)\gamma^\mu(q_1\cdot\gamma - m)\gamma^\nu], \\ T_{tu} = {} & \operatorname{tr}[(p_1\cdot\gamma - m)\gamma_\mu(q_1\cdot\gamma - m)\gamma^\nu(p_2\cdot\gamma - m)\gamma^\mu(q_2\cdot\gamma - m)\gamma_\nu]. \end{aligned} \tag{9.34}$$

The double-trace terms are most conveniently evaluated using the spinor trace formulas[10]

$$\begin{aligned} \operatorname{tr}(1) &= 4, \\ \operatorname{tr}(\gamma^\mu) &= 0, \\ \operatorname{tr}(\gamma^\mu\gamma^\nu) &= -4\eta^{\mu\nu}, \\ \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho) &= 0, \\ \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). \end{aligned} \tag{9.35}$$

This brings T_{tt} into the following form

$$\begin{aligned} T_{tt} &= 16(p_{1\mu}q_{1\nu} + q_{1\mu}p_{1\nu} - (p_1\cdot q_1 + m^2)\eta_{\mu\nu}) \\ &\qquad \cdot (p_2^\mu q_2^\nu + q_2^\mu p_2^\nu - (p_2\cdot q_2 + m^2)\eta^{\mu\nu}) \\ &= 32(p_1\cdot p_2)(q_1\cdot q_2) + 32(p_1\cdot q_2)(q_1\cdot p_2) \\ &\qquad + 32m^2(p_1\cdot q_1 + p_2\cdot q_2) + 64m^4. \end{aligned} \tag{9.36}$$

[10]The latter of these formulas follow from anti-commuting one gamma matrix past all the others.

The other double-trace term takes a similar form with q_1 and q_2 interchanged. The crossed single-trace term can be simplified by means of the enveloping identities

$$\begin{aligned}
\gamma_\mu\gamma^\mu &= -4,\\
\gamma_\mu\gamma^\nu\gamma^\mu &= 2\gamma^\nu,\\
\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu &= 4\eta^{\nu\rho},\\
\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu &= 2\gamma^\sigma\gamma^\rho\gamma^\nu.
\end{aligned}\tag{9.37}$$

After some algebra we obtain

$$\begin{aligned}
T_{tu} = &-32(q_1{\cdot}q_2)(p_1{\cdot}p_2) - 32m^4\\
&-16m^2(p_1{\cdot}p_2 + p_1{\cdot}q_1 + p_1{\cdot}q_2 + p_2{\cdot}q_1 + p_2{\cdot}q_2 + q_1{\cdot}q_2).
\end{aligned}\tag{9.38}$$

Mandelstam Invariants. In order to simplify the expressions we introduce the Mandelstam invariants

$$\begin{aligned}
s &= -(p_1+p_2)^2 = -(q_1+q_2)^2,\\
t &= -(p_1-q_1)^2 = -(p_2-q_2)^2,\\
u &= -(p_1-q_2)^2 = -(p_2-q_1)^2.
\end{aligned}\tag{9.39}$$

Inverting the relations we can write all scalar products of momenta using the s, t, u

$$\begin{aligned}
p_1{\cdot}p_2 = q_1{\cdot}q_2 &= m^2 - \tfrac{1}{2}s,\\
p_1{\cdot}q_1 = p_2{\cdot}q_2 &= \tfrac{1}{2}t - m^2,\\
p_1{\cdot}q_2 = p_2{\cdot}q_1 &= \tfrac{1}{2}u - m^2.
\end{aligned}\tag{9.40}$$

Note furthermore that momentum conservation implies the relation[11]

$$s + t + u = 4m^2.\tag{9.41}$$

Using Mandelstam invariants, the traces can be expressed very compactly as[12]

$$\begin{aligned}
T_{tt} &= 8(t^2 - 2su + 8m^4),\\
T_{uu} &= 8(u^2 - 2st + 8m^4),\\
T_{tu} &= -8(s^2 - 8m^2 s + 12m^4).
\end{aligned}\tag{9.42}$$

The squared matrix element now reads[13]

$$|M|^2 = q^4\left(\frac{u-s}{t} + \frac{t-s}{u}\right)^2 + \frac{16q^4m^2(5m^2-2s)}{tu}.\tag{9.43}$$

This expression is symmetric under exchange of t and u as it should because the outgoing particles are of the same kind.

[11]This constraint implies that functions of s, t, u can be written in several alternative ways much alike functions of p_1, p_2, q_1, q_2 which are constrained by $p_1 + p_2 - q_1 - q_2 = 0$.

[12]It is not straight-forward to derive these particular expressions, but it is easy to confirm that they match with some other expression upon substituting, e.g. $s = 4m^2 - t - u$.

[13]We can identify the first term as the corresponding result in scalar QED.

Angular Distribution. In order to understand the angular distribution of scattered particles we express the invariants in terms of the scattering angle θ

$$s = 4p^2 + 4m^2, \quad t = -2p^2(1 - \cos\theta), \quad u = -2p^2(1 + \cos\theta). \tag{9.44}$$

and insert everything into the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{q^4}{64\pi^2 e^2}\left(\frac{(4p^2 + 2m^2)^2}{p^4 \sin^4\theta} - \frac{8p^4 + 12m^2p^2 + 3m^4}{p^4 \sin^2\theta} + 1\right). \tag{9.45}$$

This expression is the leading non-trivial contribution to the angular distribution of scattered electrons.

We can however notice a problem by inspecting the expression. It diverges when

- the electron momenta p are small or
- the scattering angle θ is close to 0 or π.

In those regimes the formula cannot be trusted. The deeper reason for the divergences is that the photons which transmit the electromagnetic force are massless. Massless particles cause some conceptual problems in scattering processes.

The divergences are also relevant to the total cross section[14]

$$\sigma = \int_0^1 2\pi d\cos\theta \, \frac{d\sigma}{d\Omega}. \tag{9.46}$$

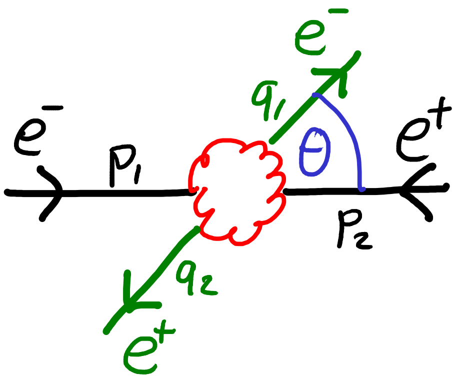
This integral diverges at $\cos\theta = 1$.

In order to properly address the above problematic regimes, one would have to take higher perturbative corrections and competing processes into account. However, only the full non-perturbative expression can provide exact results in those regimes.

Nevertheless one should not expect a meaningful result for the total cross section because the electromagnetic force is long-ranged: The photon propagator is not exponentially suppressed at long distances. Effectively all particles scatter at least by tiny amount and therefore the overall probability for scattering is 1. The scattering cross section σ is the complete area A of the bunches which is infinite due to our assumption of exactly defined momenta.

[14]The outgoing particles are indistinguishable, hence the integration extends only over one half of the spherical angles. This is sufficient since at leading order $\langle \mathrm{f}|\mathrm{i}\rangle$ has two terms one of which covers the opposite angles $\pi - \theta$. Alternative the integral over all spherical angles must be multiplied by a factor of $\frac{1}{2}$.

Crossing Symmetry. A closely related process is the scattering of electrons and positrons (Bhabha scattering).



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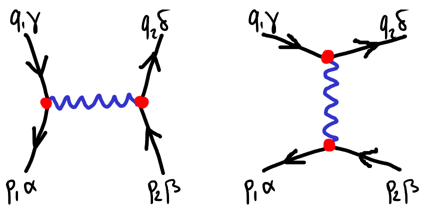
It can be computed in much the same way.

The relevant connected diagrams for electron-positron scattering are



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The cross section turns out to be exactly the same as for electron scattering but with s and u interchanged

$$s \leftrightarrow u. \tag{9.49}$$

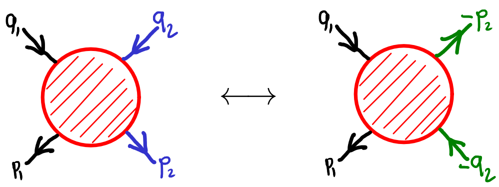
The resulting leading contribution to the squared matrix element is[15]

$$|M|^2 = q^4 \left(\frac{s-u}{t} + \frac{t-u}{s} \right)^2 + \frac{16q^4m^2(5m^2-2u)}{st}\,. \tag{9.50}$$

Indeed the computation is exactly the same when replacing

$$p_2 \leftrightarrow -q_2, \qquad \sum_\alpha v_\alpha(\vec{p}_2)\bar{v}_\alpha(\vec{p}_2) \leftrightarrow -\sum_\alpha u_\alpha(\vec{q}_2)\bar{u}_\alpha(\vec{q}_2). \tag{9.51}$$

This relationship is called crossing symmetry. In terms of Feynman diagrams, the positron line is equivalent to the electron line in reverse direction



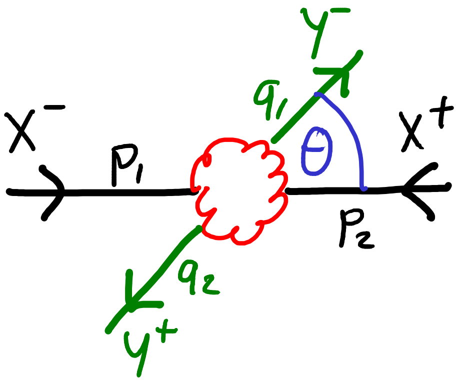
. (9.52)

[15] Apart from effects due to identical particles, the electron-positron scattering cross section does not differ substantially from the case of electron-electron scattering. The difference between attraction and repulsion manifests in the phase of matrix elements rather than in their absolute value.

9.4 Pair Production

The above electron-positron scattering involves a process where the two particles combine into a photon and subsequently split up into a pair. This process is mixed with photon exchange in the t channel.

A class of similar processes is pair production where the oppositely charged particles annihilate and create a pair of charged particles of a different kind.

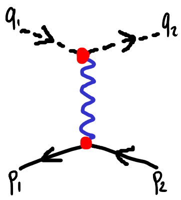


(9.53)

Let us compute scattering cross sections for such processes.

We assume that the outgoing particles have a mass m_f and charge $\pm q_\mathrm{f}$ which is different from the ingoing ones labelled by m_i and $\pm q_\mathrm{i}$.

Spinor Processes. First we consider the case where all external particles are spinors



(9.54)

There is now only one spinor trace to be evaluated

$$\begin{aligned}
T = {}& \mathrm{tr}[(p_1{\cdot}\gamma - m_\mathrm{i})\gamma_\mu(p_2{\cdot}\gamma + m_\mathrm{i})\gamma_\nu] \\
& \cdot \mathrm{tr}[(q_1{\cdot}\gamma + m_\mathrm{f})\gamma^\mu(q_2{\cdot}\gamma - m_\mathrm{f})\gamma^\nu], \\
= {}& 16(-p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu} - (-p_1{\cdot}p_2 + m_\mathrm{i}^2)\eta_{\mu\nu}) \\
& \cdot (-q_1^\mu q_2^\nu - q_2^\mu q_1^\nu - (-q_1{\cdot}q_2 + m_\mathrm{f}^2)\eta^{\mu\nu}) \\
= {}& 4(t-u)^2 + 4s^2 + 16(m_\mathrm{i}^2 + m_\mathrm{f}^2)s.
\end{aligned} \tag{9.55}$$

The Mandelstam invariants are defined as above, but due to the different masses their relationships have to be adjusted

$$\begin{aligned}
p_1{\cdot}p_2 &= m_\mathrm{i}^2 - \tfrac{1}{2}s, \\
q_1{\cdot}q_2 &= m_\mathrm{f}^2 - \tfrac{1}{2}s, \\
p_1{\cdot}q_1 = p_2{\cdot}q_2 &= \tfrac{1}{2}t - \tfrac{1}{2}m_\mathrm{i}^2 - \tfrac{1}{2}m_\mathrm{f}^2, \\
p_1{\cdot}q_2 = p_2{\cdot}q_1 &= \tfrac{1}{2}u - \tfrac{1}{2}m_\mathrm{i}^2 - \tfrac{1}{2}m_\mathrm{f}^2, \\
s + t + u &= 2m_\mathrm{i}^2 + 2m_\mathrm{f}^2.
\end{aligned} \tag{9.56}$$

Next we express the invariants as functions of the scattering angle

$$s = 4e^2, \quad t = -2p_\mathrm{i}p_\mathrm{f}(1-\cos\theta), \quad u = -2p_\mathrm{i}p_\mathrm{f}(1+\cos\theta). \tag{9.57}$$

The unpolarised matrix element squared now reads

$$\begin{aligned}|M|^2 &= \frac{q_\mathrm{i}^2 q_\mathrm{f}^2 T}{4s} \\ &= q_\mathrm{i}^2 q_\mathrm{f}^2 \left(\frac{e^2-m_\mathrm{i}^2}{e^2}\,\frac{e^2-m_\mathrm{f}^2}{e^2}\cos^2\theta + \frac{m_\mathrm{i}^2+m_\mathrm{f}^2}{e^2} + 1 \right).\end{aligned} \tag{9.58}$$

The formula for the differential cross section for our configuration of masses and momenta in the centre of mass frame reads

$$\frac{d\sigma}{d\Omega} = \sqrt{\frac{e^2-m_\mathrm{f}^2}{e^2-m_\mathrm{i}^2}}\,\frac{|M|^2}{256\pi^2 e^2}\,. \tag{9.59}$$

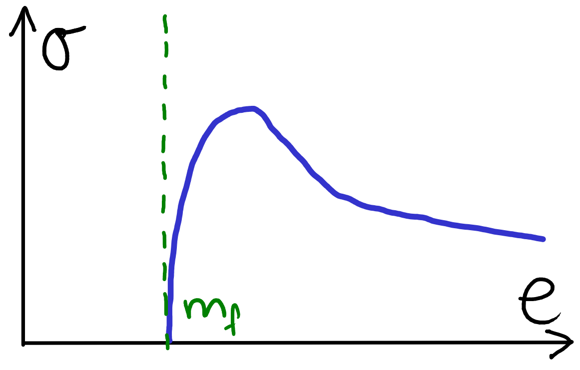
Total Cross Section. This expression is free from singularities and can be integrated to a total cross section

$$\sigma = 4\pi \int_{-1}^{1} \frac{d\cos\theta}{2}\,\frac{d\sigma}{d\Omega} = \frac{1}{64\pi e^2}\sqrt{\frac{e^2-m_\mathrm{f}^2}{e^2-m_\mathrm{i}^2}} \int_{-1}^{1} \frac{d\cos\theta}{2}\,|M|^2\,. \tag{9.60}$$

Upon integration we obtain the final result

$$\sigma = \frac{q_\mathrm{i}^2 q_\mathrm{f}^2}{48\pi e^2}\sqrt{\frac{e^2-m_\mathrm{f}^2}{e^2-m_\mathrm{i}^2}}\,\frac{e^2+\frac{1}{2}m_\mathrm{i}^2}{e^2}\,\frac{e^2+\frac{1}{2}m_\mathrm{f}^2}{e^2}\,. \tag{9.61}$$

We can plot the energy-dependence of this function.



(9.62)

Quite clearly the total energy $2e$ of the scattered particles must be at least as large as the sum of masses $2m_\mathrm{f}$ of produced particles. There is a sharp increase above production threshold, a maximum slightly above threshold (for $m_\mathrm{i} < m_\mathrm{f}$), and a slow $1/e^2$ descent.

Processes Involving Scalars. It is interesting to compare this process to the corresponding one of charged scalars. The matrix element reads

$$|M|^2 = q_\mathrm{i}^2 q_\mathrm{f}^2 \frac{(t-u)^2}{s^2} = q_\mathrm{i}^2 q_\mathrm{f}^2 \, \frac{e^2 - m_\mathrm{i}^2}{e^2} \, \frac{e^2 - m_\mathrm{f}^2}{e^2} \, \cos^2\theta\,, \tag{9.63}$$

and after integration we obtain the total cross section

$$\sigma = \frac{q_\mathrm{i}^2 q_\mathrm{f}^2}{192\pi e^2} \sqrt{\frac{e^2 - m_\mathrm{f}^2}{e^2 - m_\mathrm{i}^2}} \, \frac{e^2 - m_\mathrm{i}^2}{e^2} \, \frac{e^2 - m_\mathrm{f}^2}{e^2}\,. \tag{9.64}$$

Let us finally consider a mixed process of spinors scattering into scalars for which the matrix element reads

$$\begin{aligned} |M|^2 &= q_\mathrm{i}^2 q_\mathrm{f}^2 \frac{-(t-u)^2 + s^2 - 4m_\mathrm{f}^2 s}{s^2} \\ &= q_\mathrm{i}^2 q_\mathrm{f}^2 \, \frac{e^2 - (e^2 - m_\mathrm{i}^2)\cos^2\theta}{e^2} \, \frac{e^2 - m_\mathrm{f}^2}{e^2}\,. \end{aligned} \tag{9.65}$$

For the total cross section we obtain

$$\sigma = \frac{q_\mathrm{i}^2 q_\mathrm{f}^2}{192\pi e^2} \sqrt{\frac{e^2 - m_\mathrm{f}^2}{e^2 - m_\mathrm{i}^2}} \, \frac{e^2 + \frac{1}{2} m_\mathrm{i}^2}{e^2} \, \frac{e^2 - m_\mathrm{f}^2}{e^2}\,. \tag{9.66}$$

The opposite process of scalars scattering into spinors merely has a different overall factor

$$\sigma = \frac{q_\mathrm{i}^2 q_\mathrm{f}^2}{48\pi e^2} \sqrt{\frac{e^2 - m_\mathrm{f}^2}{e^2 - m_\mathrm{i}^2}} \, \frac{e^2 - m_\mathrm{i}^2}{e^2} \, \frac{e^2 + \frac{1}{2} m_\mathrm{f}^2}{e^2}\,. \tag{9.67}$$

We observe that, although the matrix elements differ substantially, the final cross sections take a very predictable form: There are particular factors for scalars and spinors in the initial and final states, namely

- $e^2 - m^2$ for ingoing scalars,
- $e^2 + \frac{1}{2} m_\mathrm{i}^2$ for ingoing spinors,
- $e^2 - m_\mathrm{f}^2$ for outgoing scalars,
- $4(e^2 + \frac{1}{2} m_\mathrm{f}^2)$ for outgoing spinors.

The square root on the other hand is a kinematical factor corresponding to the number/volume of initial and final states (phase space).

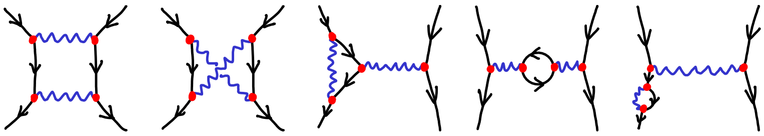
The factors actually follow from the total spin of the pairs of particles. Assume that a spin-0 state couples to the photon by a factor $e^2 - m^2$ whereas a spin-1 state couples via $e^2 + m^2$.[16] Then for scalar we immediately obtain $e^2 - m^2$ whereas the four polarisations of two spinors make up one spin-0 and three spin-1 states yielding a factor of $(e^2 - m^2) + 3(e^2 + m^2) = 4(e^2 + \frac{1}{2} m^2)$. For ingoing spinors the factor of 4 is compensated by taking the average rather than that sum.

[16]It is reasonable that close to threshold $e^2 = m^2$ the spin-1 coupling dominates because the photon is a vector particle. Above threshold the outgoing particles can also have orbital angular momentum whose spin-1 component would also couple to the photon. Therefore the increase at threshold is much softer for scalars than for spinors.

9.5 Loop Contributions

We have obtained the leading-order contributions to some particle scattering processes. Let us finally peek at contributions at higher orders in the perturbation series.

For the electron scattering process at the next order q^4 there are several types of diagrams contributing.

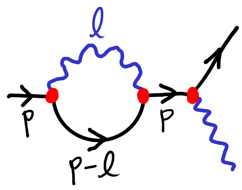


$$(9.68)$$

Here we listed only the connected diagrams up to obvious symmetric copies.

It is easy to see that these diagrams lead to two types of problems

- The diagram with a bubble on the external leg is ill-defined.



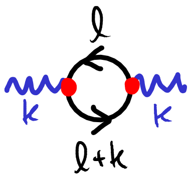
$$(9.69)$$

Due to momentum conservation the momentum on both sides of the bubble is the same. All external momenta originate from particle creation and annihilation operators $a^\dagger(\vec{p})$ and $a(\vec{p})$. These momenta are exactly on the mass shell $p^2 = -m^2$. Conversely, the internal line represents a Feynman propagator $1/(p^2 + m^2 - i\epsilon)$ which is to be evaluated right on the pole

$$\frac{1}{p^2 + m^2 - i\epsilon} \quad \text{at } p^2 = -m^2. \tag{9.70}$$

This diagram therefore makes no sense as a contribution to the scattering process.

- Most of the integrals are actually divergent in the UV, i.e. where the loop momentum ℓ is very large. For example, the bubble on the photon line yields



$$\sim \int d^4\ell \, \frac{\ell^2 + \ldots}{\ell^4 + \ldots} \sim \int \frac{d^4\ell}{\ell^2} \to \infty. \tag{9.71}$$

We have to understand how to deal with these two problems. This is going to be the subject of the final two chapters.