8 Correlation Functions

We have seen how to formally write the time evolution operator

$$U(t_1, t_0) = \mathrm{T} \exp \left(i S_{\text{int}}(t_0, t_1) \right) \tag{8.1}$$

in an interacting QFT model based on the interaction picture and time-ordered products.

A particularly convenient correlator is one where the operators are already in proper time order

$$\langle X[\phi] \rangle := \langle 0 | \mathrm{T}\big(X[\phi] \big) | 0 \rangle. \tag{8.2}$$

Such time-ordered correlation functions have multiple applications in QFT, for example, it can be used for particle scattering processed. In this chapter we will develop methods to compute them in more practical terms. The outcome will be a set graphical rules, the Feynman rules.

For simplicity we will drop all free field indices $\phi_0 \to \phi$ from now on and instead mark interacting correlators by an index "int".

8.1 Interacting Time-Ordered Correlators

Consider the correlator of two time-ordered fields with $t_1 > t_2$

$$F = \langle \phi(t_1, \vec{x}_1) \phi(t_2, \vec{x}_2) \rangle_{\text{int}} = \langle 0_{\text{int}} | \phi_{\text{int}}(t_1, \vec{x}_1) \phi_{\text{int}}(t_2, \vec{x}_2) | 0_{\text{int}} \rangle. \tag{8.3}$$

In the expression in terms of free fields

$$X = U(T, t_2) \phi(x_2) U(t_2, t_1) \phi(x_1) U(t_1, -T), \tag{8.4}$$

we notice that all operators are in proper time order and we can extend the time ordering over all the operators

$$\begin{aligned} X = {} & \left(\mathrm{T} \exp \left(i S_{\text{int}}(t_2, T) \right) \right) \phi(x_2) \left(\mathrm{T} \exp \left(i S_{\text{int}}(t_1, t_2) \right) \right) \\ & \cdot \phi(x_1) \left(\mathrm{T} \exp \left(i S_{\text{int}}(-T, t_1) \right) \right) \\ = {} & \mathrm{T}\Big(\exp \left(i S_{\text{int}}(t_2, T) \right) \phi(x_2) \exp \left(i S_{\text{int}}(t_1, t_2) \right) \\ & \quad \cdot \phi(x_1) \exp \left(i S_{\text{int}}(-T, t_1) \right) \Big). \end{aligned} \tag{8.5}$$

Inside the time-ordering symbol the order of operators does not matter. The exponents can now be combined nicely:

$$X = \mathrm{T}\Big(\phi(x_1) \phi(x_2) \exp \left(i S_{\text{int}}(-T, T) \right) \Big). \tag{8.6}$$

We thus find the correlation function $\langle \phi(x_1) \phi(x_2) \rangle_{\text{int}}$

$$F = \lim_{T \to \infty(1 - i\epsilon)} \frac{\langle 0 | \mathrm{T}\big(\phi(x_1) \phi(x_2) \exp(i S_{\text{int}}(-T, T)) \big) | 0 \rangle}{\langle 0 | \mathrm{T}\big(\exp(i S_{\text{int}}(-T, T)) \big) | 0 \rangle}. \tag{8.7}$$

This formula generalises to vacuum expectation values of arbitrary time-ordered combinations $X[\phi]$ of quantum operators

$$\begin{aligned}\langle \mathrm{T}(X[\phi])\rangle_{\text{int}} &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathrm{T}\big(X[\phi]\exp(iS_{\text{int}}(-T,T))\big)|0\rangle}{\langle 0|\mathrm{T}\big(\exp(iS_{\text{int}}(-T,T))\big)|0\rangle} \\ &\simeq \frac{\langle 0|\mathrm{T}\big(X[\phi]\exp(iS_{\text{int}})\big)|0\rangle}{\langle 0|\mathrm{T}\big(\exp(iS_{\text{int}})\big)|0\rangle}\,. \end{aligned} \tag{8.8}$$

Here the complete interaction action S_{int} implies a small imaginary part for the time coordinate in the distant past and future. We can thus express time-ordered correlators in the interacting theory in terms of similar quantities in the free theory.

This expression has several benefits and applications:

- Typically there are no ordering issues within X because time ordering puts all constituent operators into some well-defined order. This is useful when interested in the quantum expectation value of some product of classical operators.
- It directly uses the interaction terms S_{int} in the action.
- Time-ordered products and expectation values can be evaluated conveniently.
- This expression appears in many useful observables, for example in particle scattering amplitudes.

8.2 Time-Ordered Products

We now look for a method to evaluate a time-ordered correlator of a combination of free field operators $X[\phi]$

$$\langle X[\phi]\rangle := \langle 0|\mathrm{T}\big(X[\phi]\big)|0\rangle. \tag{8.9}$$

Feynman Propagator. We start with two fields

$$G_{\mathrm{F}}(x_1,x_2) = i\langle 0|\mathrm{T}\big(\phi(x_1)\phi(x_2)\big)|0\rangle. \tag{8.10}$$

By construction and earlier results it reads

$$\begin{aligned} G_{\mathrm{F}}(x_1,x_2) &= \begin{cases} i\langle 0|\phi(x_1)\phi(x_2)|0\rangle & \text{for } t_1 > t_2, \\ i\langle 0|\phi(x_2)\phi(x_1)|0\rangle & \text{for } t_2 > t_1 \end{cases} \\ &= i\theta(t_1-t_2)\Delta_+(x_1-x_2) + i\theta(t_2-t_1)\Delta_+(x_2-x_1). \end{aligned} \tag{8.11}$$

Comparing this to the retarded propagator $G_{\mathrm{R}}(x)$

$$\begin{aligned} G_{\mathrm{F}}(x) &= i\theta(t)\Delta_+(x) + i\theta(-t)\Delta_+(-x), \\ G_{\mathrm{R}}(x) &= i\theta(t)\Delta_+(x) - i\theta(t)\Delta_+(-x), \end{aligned} \tag{8.12}$$

we can write

$$G_{\mathrm{R}}(x) = G_{\mathrm{F}}(x) - i\Delta_+(-x). \tag{8.13}$$

As such it obeys the equation of a propagator,

$$-\partial^2 G_{\text{F}}(x) + m^2 G_{\text{F}}(x) = \delta^{d+1}(x), \tag{8.14}$$

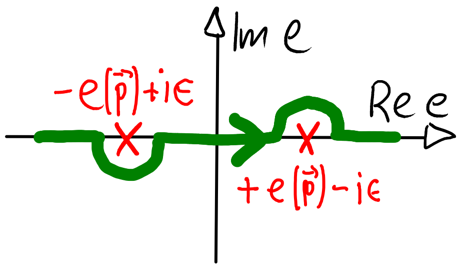
but with different boundary conditions than of the retarded propagator. It is called the Feynman propagator.

The momentum space representation of the Feynman propagator for the scalar field reads

$$G_{\text{F}}(p) = \frac{1}{p^2 + m^2 - i\epsilon}\,. \tag{8.15}$$

Here the two poles at $e = \pm e(\vec{p})$ are shifted up and down into the complex plane by a tiny amount

$$G_{\text{F}}(p) = \frac{1}{2e(\vec{p})}\left(\frac{1}{e - (-e(\vec{p}) + i\epsilon)} - \frac{1}{e - (+e(\vec{p}) - i\epsilon)}\right). \tag{8.16}$$



(8.17)

Concerning the relation to the position space representation:

- The positive energy pole $e = e(\vec{p}) - i\epsilon$ is below the real axis and thus relevant to positive times.
- The negative energy pole $e = -e(\vec{p}) + i\epsilon$ is above the real axis and thus relevant to negative times.

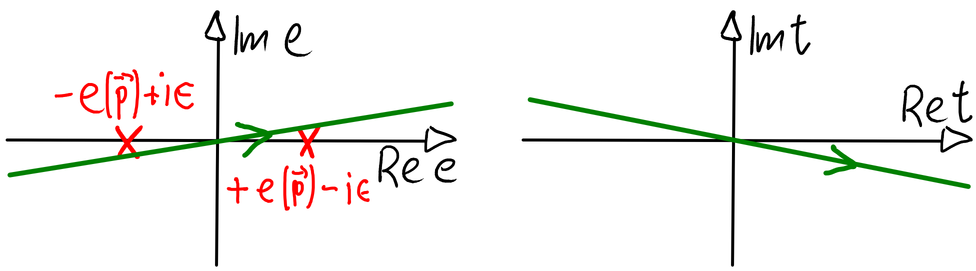
Alternatively, to obtain the correct contour around the two poles, we could integrate on a slightly tilted energy axis in the complex plane

$$e \sim (1 + i\epsilon). \tag{8.18}$$

Note that this corresponds to assuming times to be slightly imaginary, but in the opposite direction such that et is real

$$t \sim (1 - i\epsilon). \tag{8.19}$$

The $i\epsilon$ prescription of the Feynman propagators is thus directly related and equivalent to the $i\epsilon$ prescription for converting the free vacuum to the interacting one.



(8.20)

Wick's Theorem. To evaluate more complex time-ordered vacuum expectation values one typically employs Wick's theorem. It relates a time-ordered product of operators $\mathrm{T}(X[\phi])$ to a normal-ordered product of operators $\mathrm{N}(X[\phi])$. The normal-ordered product is useful when evaluating vacuum expectation values because the VEV picks out field-independent contributions only.

Let us recall the definition of normal ordering: Split up the free fields ϕ into pure creation operators ϕ^+ and pure annihilation operators ϕ^-

$$\phi = \phi^+ + \phi^-, \qquad \phi^+ \sim a^\dagger, \quad \phi^- \sim a. \tag{8.21}$$

Normal ordering of a product is defined such that all factors of ϕ^+ are to the left of all factors ϕ^-. For example,

$$\begin{aligned}\mathrm{N}\big(\phi(x_1)\phi(x_2)\big) &= \phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2)\\ &\quad + \phi^+(x_1)\phi^-(x_2) + \phi^+(x_2)\phi^-(x_1),\end{aligned} \tag{8.22}$$

where the latter two terms are in normal order and the ordering of the former two terms is irrelevant.

In comparison, time-ordering of the same product is defined as

$$\begin{aligned}\mathrm{T}\big(\phi(x_1)\phi(x_2)\big) &= \phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2)\\ &\quad + \theta(t_2-t_1)\big(\phi^+(x_2)\phi^-(x_1) + \phi^-(x_2)\phi^+(x_1)\big)\\ &\quad + \theta(t_1-t_2)\big(\phi^+(x_1)\phi^-(x_2) + \phi^-(x_1)\phi^+(x_2)\big).\end{aligned} \tag{8.23}$$

The difference between the two expressions reads

$$\begin{aligned}(\mathrm{T}-\mathrm{N})\big(\phi(x_1)\phi(x_2)\big) &= \theta(t_2-t_1)[\phi^-(x_2),\phi^+(x_1)]\\ &\quad + \theta(t_1-t_2)[\phi^-(x_1),\phi^+(x_2)]\\ &= \theta(t_2-t_1)\mathit{\Delta}_+(x_2-x_1)\\ &\quad + \theta(t_1-t_2)\mathit{\Delta}_+(x_1-x_2)\\ &= -iG_F(x_1-x_2).\end{aligned} \tag{8.24}$$

Wick's theorem is a generalisation of this result to an arbitrary number of fields: It states that the time-ordered product of a set of fields equals the partially contracted normal-ordered products summed over multiple contractions between pairs of fields. A Wick contraction between two, not necessarily adjacent, fields $\phi(x_k)$ and $\phi(x_l)$ replaces the relevant two field operators by their Feynman propagator $-iG_\mathrm{F}(x_k, x_l)$, in short:

$$\begin{aligned}&[\ldots\phi_{k-1}\underbrace{\phi_k\phi_{k+1}\ldots\phi_{l-1}\phi_l}\phi_{l+1}\ldots]\\ &:= -iG_\mathrm{F}(x_1-x_2)\,[\ldots\phi_{k-1}\;\phi_{k+1}\ldots\phi_{l-1}\;\phi_{l+1}\ldots].\end{aligned} \tag{8.25}$$

For example:

$$
\begin{aligned}
\mathrm{T}\big(\phi_1\phi_2\big) &= \mathrm{N}\big(\phi_1\phi_2\big) + \wick{\c1\phi_1\c1\phi_2},\\
\mathrm{T}\big(\phi_1\phi_2\phi_3\big) &= \mathrm{N}\big(\phi_1\phi_2\phi_3\big) + \wick{\c1\phi_1\c1\phi_2\phi_3} + \wick{\c1\phi_1\phi_2\c1\phi_3} + \wick{\phi_1\c1\phi_2\c1\phi_3},\\
\mathrm{T}\big(\phi_1\phi_2\phi_3\phi_4\big) &= \mathrm{N}\Big[\phi_1\phi_2\phi_3\phi_4 + \wick{\c1\phi_1\c1\phi_2\phi_3\phi_4} + \wick{\c1\phi_1\phi_2\c1\phi_3\phi_4}\\
&\qquad + \wick{\c1\phi_1\phi_2\phi_3\c1\phi_4} + \wick{\phi_1\c1\phi_2\c1\phi_3\phi_4}\\
&\qquad + \wick{\phi_1\c1\phi_2\phi_3\c1\phi_4} + \wick{\phi_1\phi_2\c1\phi_3\c1\phi_4}\Big]\\
&\quad + \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4},
\end{aligned}
\tag{8.26}
$$

To prove the statement by induction is straight-forward:

- Assume the statement holds for $n-1$ fields.
- Arrange n fields in proper time order $\phi_n\ldots\phi_1$ with $t_n > \ldots > t_1$.
- Consider $\mathrm{T}[\phi_n\ldots\phi_1] = (\phi_n^+ + \phi_n^-)\mathrm{T}[\phi_{n-1}\ldots\phi_1]$ and replace $\mathrm{T}[\phi_{n-1}\ldots\phi_1]$ by contracted normal-ordered products.
- ϕ_n^+ is already in normal order, it can be pulled into $\mathrm{N}[\ldots]$.
- Commute ϕ_n^- past all the remaining fields in the normal ordering.
- For every uncontracted field ϕ_k in $\mathrm{N}[\ldots]$, pick up a term $\Delta_+(x_n - x_k) = -iG_{\mathrm{F}}(x_n - x_k)$ because $t_n > t_k$.
- Convince yourself that all contractions of n fields are realised with unit weight.
- Convince yourself that for different original time-orderings of $\phi_n\ldots\phi_1$, the step functions in G_{F} do their proper job.

Time-Ordered Correlators. To compute time-ordered correlators we can use the result of Wick's theorem. All the normal-ordered terms with remaining fields drop out of vacuum expectation values. The only terms to survive are those where all the fields are complete contracted in pairs

$$
\langle\phi_1\ldots\phi_n\rangle := \sum_{\substack{\text{complete}\\ \text{contractions}}} \wick{\c1\phi_1\c2\phi_2 \ldots \c2{\cdot} \ldots \c3{\cdot}\ldots\c3{\cdot}\c4{\cdot}\ldots \c4\phi_{n-1}\c1\phi_n}.
\tag{8.27}
$$

In particular, it implies that correlators of an odd number of fields must be zero.

This formula applies directly to a single species of real scalar fields, but for all the other fields and mixed products there are a straight-forward equivalents:

- For fields with spin, use the appropriate propagator, e.g. $(G^{\mathrm{D}})^a{}_b$ for contracting the Dirac fields ψ^a and $\bar{\psi}_b$.
- For any crossing of lines attached to fermionic fields, multiply by a factor of (-1).

8.3 Some Examples

We have learned how to reduce time-ordered correlators in a weakly interacting QFT to free time-ordered correlators

$$\langle X[\phi]\rangle_{\text{int}} = \frac{\langle X[\phi]\ \exp(iS_{\text{int}}[\phi])\rangle}{\langle \exp(iS_{\text{int}}[\phi])\rangle}\,. \tag{8.28}$$

We have also learned how to evaluate the latter

$$\langle \phi_1 \ldots \phi_n\rangle := \sum_{\substack{\text{complete}\\ \text{contractions}}} \phi_1\phi_2 \ldots\ldots\ldots\ldots \phi_{n-1}\phi_n. \tag{8.29}$$

We will now apply these formulas to some basic types of time-ordered correlators in order to develop an understanding for them.

Setup. We will consider ϕ^4 theory, i.e. a single real scalar field with a ϕ^4 interaction

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\lambda\phi^4. \tag{8.30}$$

We define the interaction picture using the quadratic terms in the action

$$\mathcal{L}_0 = -\tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2. \tag{8.31}$$

What remains is the interaction term

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{24}\lambda\phi^4, \tag{8.32}$$

whose coefficient, the coupling constant λ, is assumed to be small. The interaction part of the action is thus

$$S_{\text{int}}(t_1, t_2) := \int_{t_1}^{t_2} dt \int d^3\vec{x}\, \mathcal{L}_{\text{int}}(x), \quad S_{\text{int}} := S_{\text{int}}(-\infty, +\infty). \tag{8.33}$$

We would like to evaluate the correlators of two and four fields

$$T_{12} = \langle\phi_1\phi_2\rangle_{\text{int}}, \qquad F_{1234} = \langle\phi_1\phi_2\phi_3\phi_4\rangle_{\text{int}}, \tag{8.34}$$

where ϕ_k denotes the field $\phi(x_k)$ evaluated at position x_k. These are functions of the coupling constant λ which we formally expand for small λ as

$$T(\lambda) = \sum_{n=0}^{\infty} T^{(n)}, \quad F(\lambda) = \sum_{n=0}^{\infty} F^{(n)}, \qquad T^{(n)} \sim F^{(n)} \sim \lambda^n. \tag{8.35}$$

Leading Order. First, we shall evaluate T and F at lowest order in the coupling strength. At leading order we simply set $\lambda = 0$ and obtain the correlator in the free theory

$$T^{(0)} = \langle\phi_1\phi_2\rangle, \qquad F^{(0)} = \langle\phi_1\phi_2\phi_3\phi_4\rangle. \tag{8.36}$$

Using Wick's theorem this evaluates to

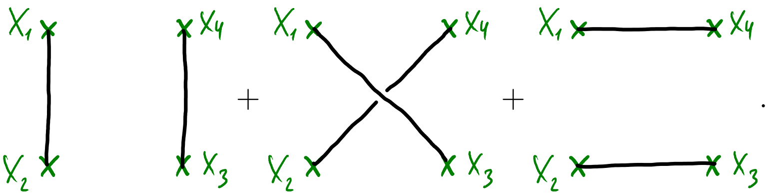
$$\begin{aligned} T^{(0)} &= \underbracket{\phi_1\phi_2}, \\ F^{(0)} &= \underbracket{\phi_1\phi_2}\,\underbracket{\phi_3\phi_4} + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4. \end{aligned} \tag{8.37}$$

More formally, these equal

$$\begin{aligned} T^{(0)} &= (-i)G_{12}, \\ F^{(0)} &= (-i)^2 G_{12}G_{34} + (-i)^2 G_{13}G_{24} + (-i)^2 G_{14}G_{23}, \end{aligned} \tag{8.38}$$

where G_{kl} denotes $G_{\mathrm{F}}(x_k - x_l)$. In a graphical notation we could write this as

$T^{(0)} =$ ,

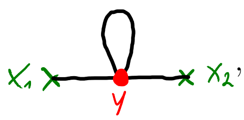
$F^{(0)} =$  . (8.39)

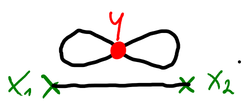
Each vertex represents a spacetime point x_k and each line connecting two vertices k and l represents a propagator $-iG_{\mathrm{F}}(x_k - x_l)$.

Two-Point Function at First Order. The contributions to the interacting two-point function at the next perturbative order read

$$\begin{aligned} T^{(1)} &= \langle\phi_1\phi_2 iS_{\mathrm{int}}[\phi]\rangle - \langle\phi_1\phi_2\rangle\langle iS_{\mathrm{int}}[\phi]\rangle \\ &= -\frac{i\lambda}{24}\int d^4y\, \langle\phi_1\phi_2\phi_y\phi_y\phi_y\phi_y\rangle \\ &\quad + \frac{i\lambda}{24}\int d^4y\, \langle\phi_1\phi_2\rangle\langle\phi_y\phi_y\phi_y\phi_y\rangle. \end{aligned} \tag{8.40}$$

Using Wick's theorem the two terms expand to 15 and 3 contributions. Consider the first term only: The 15 contributions can be grouped into two types. The first type receives 12 identical contributions from contracting the 4 identical ϕ_y's in superficially different ways. The remaining 3 terms in the second group are identical for the same reason. We summarise the groups as follows

$T^{(1)}_{1\mathrm{a}} = -\frac{i}{2}\lambda\int d^4y\, \phi_1\phi_2\phi_y\phi_y\underbracket{\phi_y\phi_y} = -\frac{i}{2}\lambda\int d^4y$ ,

$T^{(1)}_{1\mathrm{b}} = -\frac{i}{8}\lambda\int d^4y\, \underbracket{\phi_1\phi_2}\,\underbracket{\phi_y\phi_y}\,\underbracket{\phi_y\phi_y} = -\frac{i}{8}\lambda\int d^4y$ . (8.41)

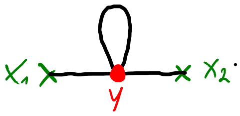
The term originating from the denominator of the interacting correlator evaluates to

$$T_2^{(1)} = -\tfrac{1}{8}(-i\lambda)\int d^4y\, \phi_1\phi_2\phi_y\phi_y\phi_y\phi_y = -T_{1\text{b}}^{(1)}. \tag{8.42}$$

It precisely cancels the second contribution to the first term. Altogether we find the following expression for the first-order correction to the two-point function

$$T^{(1)} = T_{1\text{a}}^{(1)} = \tfrac{1}{2}(-i)^4\lambda \int d^4y\, G_{1y}G_{2y}G_{yy}$$

$$= -\tfrac{i}{2}\lambda \int d^4y \tag{8.43}$$



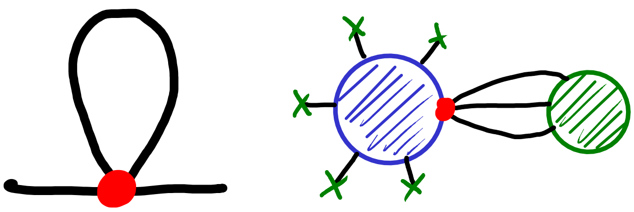
Tadpoles. We were careful enough not to write this expression too explicitly

$$T^{(1)} = \tfrac{1}{2}\lambda\, G_{\text{F}}(0)\int d^4y\, G_{\text{F}}(x_1-y)G_{\text{F}}(x_2-y). \tag{8.44}$$

We notice that one of the propagators decouples from the function. Moreover its argument is precisely zero because the propagator connects a point to itself.

- The result is in general divergent, it is very similar to the vacuum energy we encountered much earlier in QFT.
- In our derivation of time ordering we were sloppy in that we did not discuss the case of equal times. In a local Lagrangian, however, all terms are defined at equal time, moreover at equal spatial position. It would make sense to employ normal ordering in this case, which eliminates the term at the start.
- Whatever the numerical value of $G_{\text{F}}(0)$, even if infinite, it does not yield any interesting functional dependence to $T^{(1)}$. In fact it could be eliminated by adding a term $-\tfrac{i}{4}\lambda G_{\text{F}}(0)\phi^2$ to the interaction Lagrangian. This has the same effect as normal ordering the Lagrangian.

This term is called a tadpole term because the corresponding diagram looks like a tadpole sitting on the propagator line



(8.45)

More generally, tadpoles are internal parts of a diagram which are attached to the rest of the diagram only via a single vertex. In most cases, they can be compensated by adding suitable local terms to the interaction Lagrangian. Even though this correction term is somewhat dangerous and somewhat trivial, let us pretend it is a regular contribution and carry it along.

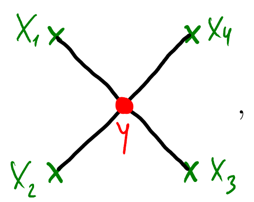
First Order Four-Point Function. The first-order contributions to the interacting four-point function take a similar form

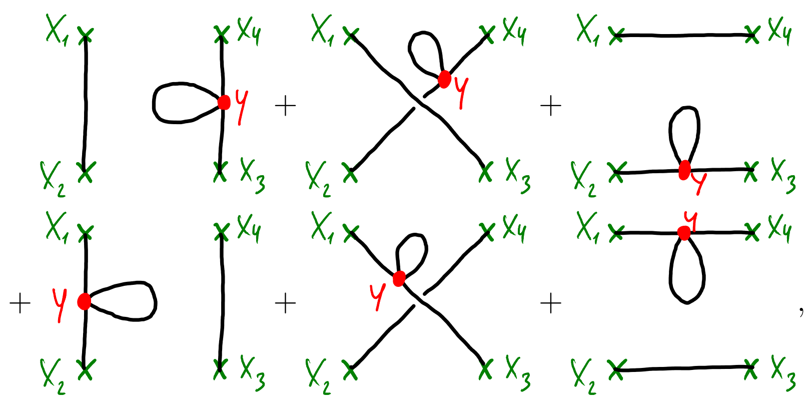
$$\begin{aligned} F^{(1)} &= \langle \phi_1\phi_2\phi_3\phi_4 iS_{\text{int}}[\phi]\rangle - \langle \phi_1\phi_2\phi_3\phi_4\rangle\langle iS_{\text{int}}[\phi]\rangle \\ &= -\frac{i\lambda}{24}\int d^4y\, \langle \phi_1\phi_2\phi_3\phi_4\phi_y\phi_y\phi_y\phi_y\rangle \\ &\quad + \frac{i\lambda}{24}\int d^4y\, \langle \phi_1\phi_2\phi_3\phi_4\rangle\langle\phi_y\phi_y\phi_y\phi_y\rangle. \end{aligned} \tag{8.46}$$

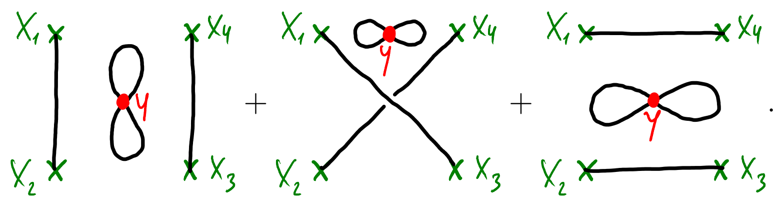
These expressions are not as innocent as they may look: Using Wick's theorem the two terms expand to $7\cdot 5\cdot 3\cdot 1 = 105$ and $3\cdot 3 = 9$ terms. Gladly, most of these terms are identical and can be summarised, we group them into 24, $6\cdot 12$ and $3\cdot 3$ terms from the first contribution and $3\cdot 3$ terms from the second one

$$\begin{aligned} F^{(1)}_{\text{1a}} &= (-i\lambda)\int d^4y\, \phi_1\phi_2\phi_3\phi_4\phi_y\phi_y\phi_y\phi_y, \\ F^{(1)}_{\text{1b}} &= \tfrac{1}{2}(-i\lambda)\int d^4y\, \phi_1\phi_2\phi_3\phi_4\phi_y\phi_y\phi_y\phi_y + 5\text{ perm.}, \\ F^{(1)}_{\text{1c}} &= \tfrac{1}{8}(-i\lambda)\int d^4y\, \phi_1\phi_2\phi_3\phi_4\phi_y\phi_y\phi_y\phi_y + 2\text{ perm.}, \\ F^{(1)}_{2} &= -\tfrac{1}{8}(-i\lambda)\int d^4y\, \phi_1\phi_2\phi_3\phi_4\phi_y\phi_y\phi_y\phi_y + 2\text{ perm..} \end{aligned} \tag{8.47}$$

The graphical representation of these terms is

$F^{(1)}_{\text{1a}} \simeq$  ,

$F^{(1)}_{\text{1b}} \simeq$  ,

$F^{(1)}_{\text{1c}} \simeq$  . (8.48)

Let us now discuss the roles of the three terms.

Vacuum Bubbles. In the above result we notice that again the contribution $F_2^{(1)}$ from the denominator of the interacting correlator cancels the term $F_{1c}^{(1)}$ from the numerator. This effect is general:

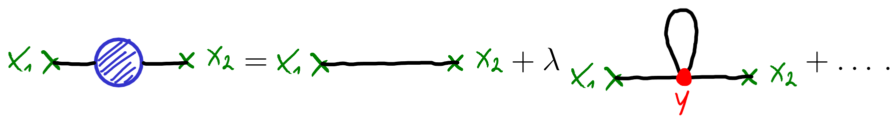
Some graphs have components which are coupled neither to the rest of the graph nor to the external points. Such parts of the graph are called vacuum bubbles.

- Vacuum bubbles represent some virtual particles which pop out of the quantum mechanical vacuum and annihilate among themselves. They do not interact with any of the physically observed particles, hence one should be able to ignore such contributions.
- Vacuum bubbles are usually infinite. Here we obtain as coefficient $G_{\mathrm{F}}(0)^2 \int d^4y$. This contains two divergent factors of $G_{\mathrm{F}}(0)$ and an infinite spacetime volume $\int d^4y$.
- Formally, we could remove such terms by adding a suitable field-independent term to the Lagrangian. Alternatively we could normal order it.
- In any case, vacuum bubbles are generally removed by the denominator of the interacting correlation function. This cancellation ensures that the interacting vacuum is properly normalised, $\langle 0|0\rangle_{\mathrm{int}} = 1$. Any diagram containing at least one vacuum bubble can be discarded right away.

Disconnected Graphs. The contribution from $F_{1b}^{(1)}$ is reminiscent of the correction $T^{(1)}$ to the two-point function. In fact it can be written as a sum of products of two-point functions

$$\begin{aligned} F_{1b}^{(1)} = T_{12}^{(0)}T_{34}^{(1)} + T_{13}^{(0)}T_{24}^{(1)} + T_{14}^{(0)}T_{23}^{(1)} \\ + T_{12}^{(1)}T_{34}^{(0)} + T_{13}^{(1)}T_{24}^{(0)} + T_{14}^{(1)}T_{23}^{(0)}. \end{aligned} \tag{8.49}$$

This combination is precisely the first-order contribution to a product of two $T(\lambda)$'s



(8.50)

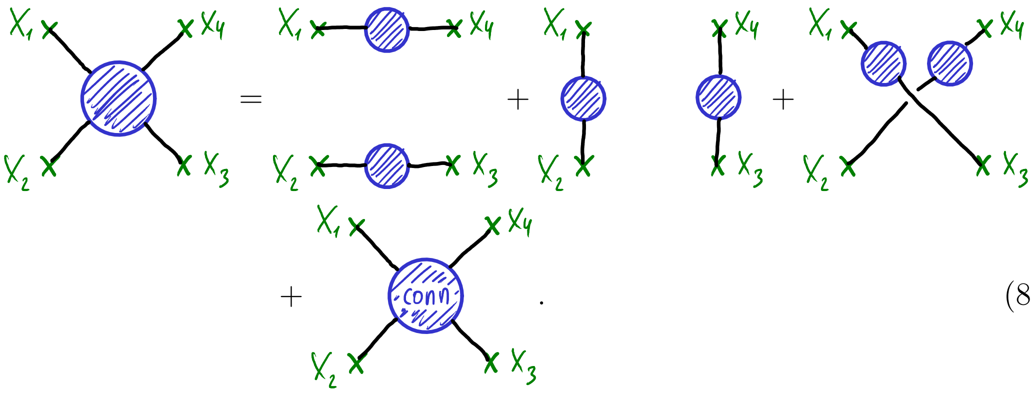
This is also a general feature of correlation functions:

- Correlation functions contain disconnected products of lower-point functions. The corresponding graphs contain disconnected components (each of which is connected to at least one external field).
- Such contributions are typically put aside because their form is predictable.[1] Nevertheless, they are essential and non-negligible contributions.
- Such disconnected contributions represent processes that take place simultaneously without interfering with each other.

[1] When computing an n-point function one will typically already have computed all the k-point functions with $k < n$ anyway.

Quite generally one can split the contributions into connected and disconnected terms. Here we know, to all orders in λ

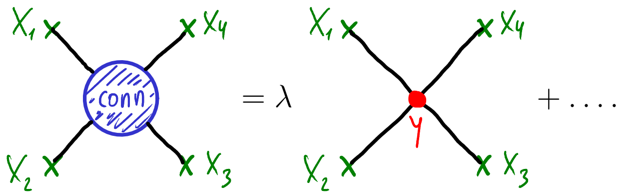
$$
\begin{aligned}
F(\lambda) = {} & T_{12}(\lambda)T_{34}(\lambda) + T_{13}(\lambda)T_{24}(\lambda) + T_{14}(\lambda)T_{23}(\lambda) \\
& + F_{\text{conn}}(\lambda)
\end{aligned}
$$



(8.51)

where $F_{\text{conn}}(\lambda)$ summarises all connected contributions. In our case

$$
F^{(1)}_{\text{conn}} = F^{(1)}_{1\text{a}} = -i\lambda \int d^4y\, G_{1y}G_{2y}G_{3y}G_{4y}
$$



(8.52)

Symmetry Factors. In our computation, we have encountered many equivalent contributions which summed up into a single term. We observe that these sums have conspired to cancel most of the prefactors of $1/24$. The purpose of having a prefactor of $1/24$ for ϕ^4 in the action is precisely to be cancelled against multiplicities in correlators, where λ typically appears without or with small denominators.

We can avoid constructing a large number of copies of the same term by considering the symmetry of terms or the corresponding graphs. The symmetry factor is the inverse size of the discrete group that permutes the elements of a term or a graph while leaving its structure invariant.

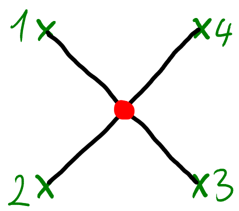
To make use of symmetry factors for the calculation of correlation functions, one should set up the Lagrangian such that every product of terms comes with the appropriate symmetry factor. For example, the term ϕ^4 allows arbitrary permutations of the 4 ϕ's. There are $4! = 24$ such permutations, hence the appropriate symmetry factor is $1/24$.[2]

The crucial insight is the following: When the symmetry factors for the Lagrangian are set up properly, the summed contributions to correlation function also have their appropriate symmetry factors.

[2]After all, we are free to call the term that multiplies ϕ^4 either $\lambda/24$ or λ'. It is not even difficult to translate between them.

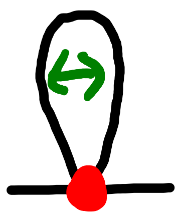
To determine the symmetry factors correctly sometimes is difficult, as one has to identify all permissible permutations. This can be difficult, for example, when the graphical representation has a difficult topology or when it hides some relevant information.

Let us consider the symmetry factors of the terms we have computed so far. The contributions $T^{(0)}$, $F^{(0)}$ and $F^{(1)}_{\text{conn}}$ have trivial symmetry factors.



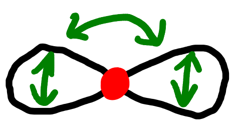
(8.53)

Permutations of any of the elements would change the labelling of the external fields. The symmetry factor for the tadpole diagram is 1/2.



(8.54)

The relevant $\mathbb{Z}_2$ symmetry flips the direction of the tadpole line.[3] Finally, the vacuum bubble diagram has a symmetry factor of 1/8.



(8.55)

There are two factors of 1/2 for flipping the direction of the tadpole lines. Then there is another factor of 1/2 for permuting the two tadpole lines.

8.4 Feynman Rules

We have seen how to evaluate some perturbative contributions to interacting correlators. Following the formal prescription leads to a lot of combinatorial overhead as the results tend to be reasonably simple compared to the necessary intermediate steps. Feynman turned the logic around and proposed a simple graphical construction of correlators:

The interacting correlator of several fields can be expressed as a sum of so-called Feynman graphs. Each Feynman graph represents a certain mathematical expression which can be evaluated from the graph by the Feynman rules. Moreover a Feynman graph display nicely the physical process that leads to the corresponding term of the correlator.

[3]In fact, the symmetry acts on the connections of lines to vertices. Here, exchanging the two endpoints of the tadpole line is the only symmetry.

For every weakly coupled QFT there is a set of Feynman rules to compute its correlators.[4] Here we list the Feynman rules for the scalar ϕ^4 model.

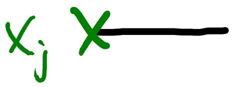
Feynman Rules in Position Space. A permissible graph for a correlator

$$F(x_1, \ldots, x_n) = \langle \phi(x_1) \ldots \phi(x_n) \rangle_{\text{int}} \tag{8.56}$$

- has undirected and unlabelled edges,

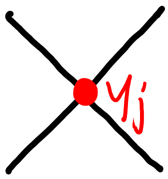
(8.57)

- has n 1-valent (external) vertices labelled by x_j,



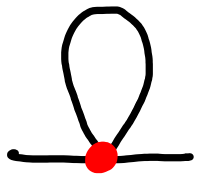
(8.58)

- has an arbitrary number k of 4-valent (internal) vertices labelled by y_j,



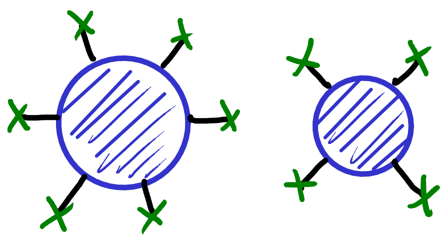
(8.59)

- can have lines connecting a vertex to itself (tadpole),



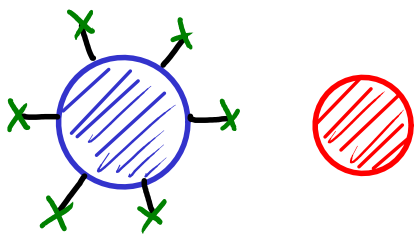
(8.60)

- can have several connection components,



(8.61)

- must not have components disconnected from all of the external vertices x_j (vacuum bubble),



(8.62)

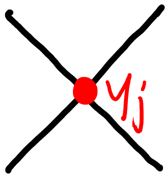
[4]Similar graphs and rules can actually be set up and applied to a wide range of algebraic problems not at all limited to relativistic QFT's.

For each topologically distinct graph we can compute a contribution according to the following rules:

- For each edge connecting two vertices z_i and z_j write a factor of $-iG_{\mathrm{F}}(z_i - z_j)$.

$$\rightarrow -iG_{\mathrm{F}}(z_i - z_j) \tag{8.63}$$

- For each 4-valent vertex y_j, write a factor of $-i\lambda$ and integrate over $\int d^4y_j$.



$$\rightarrow -i\lambda \int d^4y_j \tag{8.64}$$

- Multiply by the appropriate symmetry factor, i.e. divide by the number of discrete symmetries of the graph.

Feynman Rules in Momentum Space. One of the problems we have not yet mentioned is that the Feynman propagator G_{F} is a complicated function in spacetime. Moreover, we need to compute multiple convolution integrals of these functions over spacetime, e.g. the integral defining $F^{(1)}_{\mathrm{conn}}$. This soon enough exceeds our capabilities.

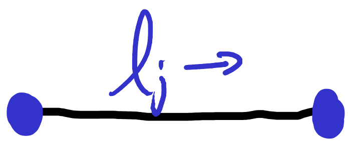
These computations can be simplified to some extent by going to momentum space. Such a momentum space representation will be particularly useful later when we compute the interaction between particles with definite momenta in particle scattering experiments.

The momentum space version is defined as follows[5]

$$F(p_1, \ldots, p_n) = \int d^4x_1 \ldots d^4x_n \, e^{ix_1 \cdot p_1 + \ldots + ix_n \cdot p_n} \langle \phi_1 \ldots \phi_n \rangle. \tag{8.65}$$

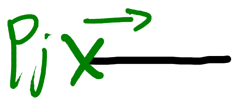
A Feynman graph in momentum space

- has edges labelled by a directed flow of 4-momentum ℓ_j from one end to the other,



(8.66)

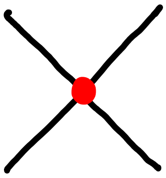
- has n 1-valent (external) vertices with an inflow of 4-momentum p_j,



(8.67)

[5]Note that we are evaluating a time-ordered correlator. This is well-defined in position space and we have to perform the Fourier integrals after computing the correlator. It implies that the momenta p_j can and should be taken off-shell $p_j^2 + m^2 \neq 0$. It is different from computing a correlator such as $\langle 0|a(\vec{p}_1) \ldots a^\dagger(\vec{p}_n)|0\rangle$ where all the momenta are defined only on shell $p_j^2 + m^2 = 0$.

- has an arbitrary number k of 4-valent (internal) vertices which conserve the flow of momentum

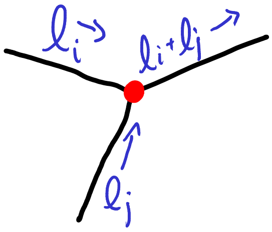


$$ \tag{8.68}$$

- shares the remaining attributes with the position space version.

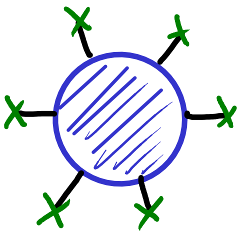
The Feynman rules for evaluating a graph read

- Work out the flow of momentum from the external vertices across the internal vertices. Label all edges with the appropriate momenta ℓ_j.



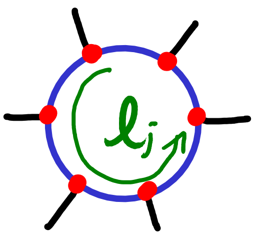
$$ \tag{8.69}$$

- There is a momentum-conservation condition $p_{j_1} + \ldots + p_{j_m}$ for each connected component of the graph. Write a factor of $(2\pi)^4\delta^4(p_{j_1} + \ldots + p_{j_m})$ including all contributing external momenta p_j.



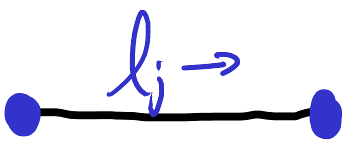
$$\to (2\pi)^4\delta^4(p_{j_1} + \ldots + p_{j_m}) \tag{8.70}$$

- For each internal loop of the graph, there is one undetermined 4-momentum ℓ_j. Integrate the final expression over all such momenta $\int d^4\ell_j/(2\pi)^4$.



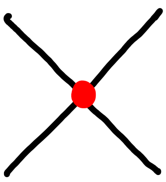
$$\to \int \frac{d^4\ell_j}{(2\pi)^4} \tag{8.71}$$

- For each edge write a factor of $-i/(\ell_j^2 + m^2 - i\epsilon)$.



$$\to \frac{-i}{\ell_j^2 + m^2 - i\epsilon} \tag{8.72}$$

- For each 4-valent vertex, write a factor of $-i\lambda$.



$$\to -i\lambda \tag{8.73}$$

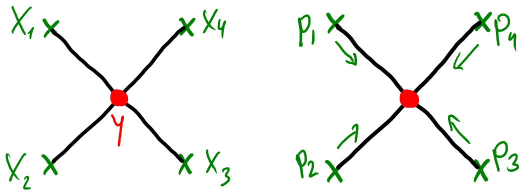
- Multiply by the appropriate symmetry factor, i.e. divide by the number of discrete symmetries of the graph.

General Models. We observe that the Feynman graphs and rules for a QFT model reflect quite directly the content of its action:

- In particular, the free part of the action S_0 determines the types and features of the fields and particles. These are reflected by the Feynman propagator G_{F} which is associated to the edges.
- The interaction part of the action S_{int} contains all the information about the set of interaction vertices.

Examples. Let us apply the Feynman rules to compute the mathematical expressions for a few Feynman graphs.

Consider first the graph for the leading connected contribution $F^{(1)}_{\mathrm{conn}}$ to the four-point function.



(8.74)

Applying the rules for position space, we obtain right away

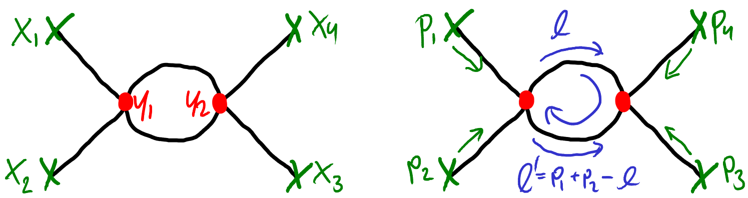
$$F^{(1)}_{\mathrm{conn}} = -i\lambda \int d^4y \prod_{j=1}^{4} G_{\mathrm{F}}(x_j - y)\,. \tag{8.75}$$

In momentum space, the corresponding result is

$$F^{(1)}_{\mathrm{conn}} = -i\lambda(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\prod_{j=1}^{4}\frac{1}{p_j^2+m^2-i\epsilon}\,. \tag{8.76}$$

This expression is merely a rational function and does not contain any integrals. It is therefore conceivably simpler than its position space analog. Unfortunately, it is generally not easy to perform the Fourier transformation to position space.[6]

Next, consider a slightly more complicated example involving an internal loop.



(8.77)

[6] A notable exception is the massless case where the correlation functions in position space has a reasonably simple form.

Evaluation of the Feynman graph in position space is straight-forward

$$F = \frac{1}{2}(-i\lambda)^2(-i)^6 \int d^4y_1\, d^4y_2\, G_{\mathrm{F}}(y_1 - y_2)^2 \cdot G_{\mathrm{F}}(x_1 - y_1)G_{\mathrm{F}}(x_2 - y_1)G_{\mathrm{F}}(x_3 - y_2)G_{\mathrm{F}}(x_4 - y_2). \tag{8.78}$$

The symmetry factor is 1/2 because the two lines of the internal loop can be interchanged.

For momentum space, we first have to label the remaining lines along the internal loop: The total flow of momentum *into* the left vertex from the external lines is $p_1 + p_2$, whereas the momenta on both internal lines are yet undetermined. The sum of internal momenta flowing into the vertex must therefore equal $-p_1 - p_2$ by momentum conservation.[7] One internal momentum remains undetermined, let us call it ℓ and eventually integrate over it. The other one must equal $\ell' = p_1 + p_2 - \ell$.

$$F = \frac{1}{2}(-i\lambda)^2(-i)^6(2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4)\prod_{j=1}^{4}\frac{1}{p_j + m^2 - i\epsilon} \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2 + m^2 - i\epsilon}\,\frac{1}{(p_1 + p_2 - \ell)^2 + m^2 - i\epsilon}\,. \tag{8.79}$$

Now we are left with a multiple integral over a rational function. There exist techniques to deal with this sort of problem, we will briefly discuss some of the basic ones at the end of this course. Some integrals like this one can be performed, but most of them remain difficult and it is an art to evaluate them. Unfortunately, numerical methods generally are not applicable either. This is a generic difficulty of QFT with no hope for a universal solution. The Feynman rules are a somewhat formal method and it is hard to extract concrete numbers or functions from them.

8.5 Feynman Rules for QED

Finally, we would like to list the Feynman rules for the simplest physically relevant QFT model, namely quantum electrodynamics (QED). We shall use the Lagrangian in Feynman gauge

$$\mathcal{L}_0 = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi - \tfrac{1}{2}\partial^\mu A^\nu\partial_\mu A_\nu, \quad \mathcal{L}_{\mathrm{int}} = q\,\bar\psi\gamma^\mu\psi\, A_\mu. \tag{8.80}$$

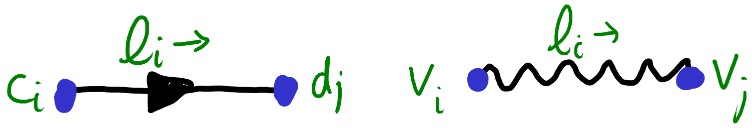
A non-trivial interacting correlation function in this model must contain as many fermionic fields ψ as conjugates $\bar\psi$ due to global U(1) symmetry. Consider such a correlation function

$$\langle A_{\mu_1}(k_1)\ldots A_{\mu_m}(k_m)\,\bar\psi_{a_1}(p_1)\psi^{b_1}(q_1)\ldots\bar\psi_{a_n}(p_n)\psi^{b_n}(q_n)\rangle. \tag{8.81}$$

Admissible Feynman graphs have the following properties in addition or instead of to the ones of the ϕ^4 model:

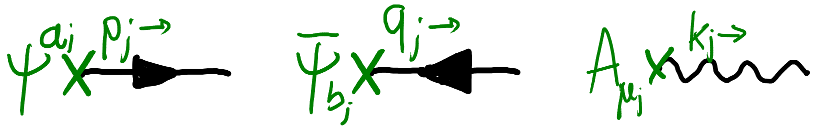
[7] By considering the right vertex, it must also equal $p_3 + p_4$. This requirement is consistent by means of overall momentum conservation $p_1 + p_2 + p_3 + p_4 = 0$.

- There are two types of edges: undirected wavy lines (photons) or directed straight lines (electrons and/or positrons).



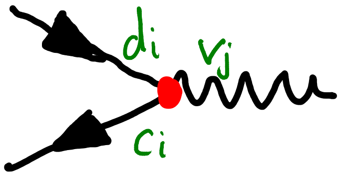
(8.82)

- The edges are labelled by a directed flow of 4-momentum ℓ_j.
- The ends of wavy lines are labelled by indices ρ_j and σ_j; the ends of straight lines are labelled by indices c_j and d_j in the direction of the arrow of the straight line.
- There is a 1-valent (external) vertex for each field in the correlator. The momentum inflow and the label at the end of the edge are determined by the corresponding field.



(8.83)

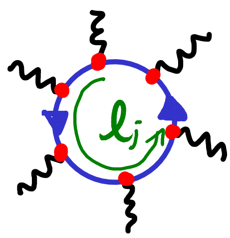
- There is one type of (internal) vertex: It is 3-valent and connects an ingoing and an outgoing straight line (fermion) with a wavy (photon) line.



(8.84)

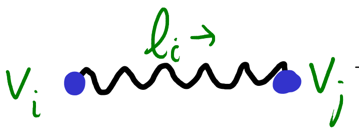
The QED-specific Feynman rules read as follows:

- The graph can have only fermion loops, which contribute an extra factor of (-1) due to their statistics.



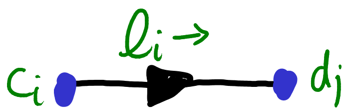
$$\to (-1) \int \frac{d^4\ell_j}{(2\pi)^4} \tag{8.85}$$

- For each wavy edge write a factor of $-i\eta_{\nu_i\nu_j}/(\ell_j^2 - i\epsilon)$.



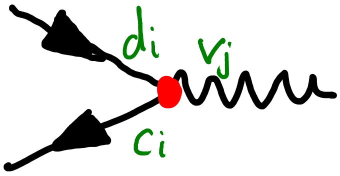
$$\to \frac{-i\eta_{\nu_i\nu_j}}{\ell_j^2 - i\epsilon} \tag{8.86}$$

for each straight edge write a factor of $-i(\ell_j{\cdot}\gamma + m)^{c_j}{}_{d_j}/(\ell_j^2 + m^2 - i\epsilon)$.



$$\to \frac{-i(\ell_j{\cdot}\gamma + m)^{c_i}{}_{d_j}}{\ell_j^2 + m^2 - i\epsilon} \tag{8.87}$$

- For each 3-valent vertex, write a factor of $-iq(\gamma^{\nu_j})^{c_i}{}_{d_i}$.



$$\to -iq(\gamma^{\nu_j})^{c_i}{}_{d_i} \tag{8.88}$$

QED and Gauge Invariance. Note that QED is a gauge theory which requires some gauge fixing. Feynman gauge is very convenient, but any other consistent gauge is acceptable, too. Different gauges imply different propagators which lead to non-unique results for correlation functions. Unique results are only to be expected when the field data within the correlation function is gauge invariant.

More precisely, the correlator should contain the gauge potential $A_\mu(x)$ only in the combination $F_{\mu\nu}(x)$ or as the coupling $\int d^4x\, J^\mu A_\mu$ to some conserved current $J_\mu(x)$. Moreover, charged spinor fields should be combined into uncharged products, e.g. $\bar\psi(x)\ldots\psi(x)$ potentially dressed with covariant derivatives.