7 Interactions

We have learned a lot about the three basic constituents of QFT in four dimensions:

- scalar fields (spin $j = 0$ or helicity $h = 0$),
- spinor fields (spin $j = \frac{1}{2}$ or helicity $h = \pm\frac{1}{2}$),
- vector fields (helicity $h = \pm 1$ or spin $j = 1$).

So far we considered only free fields. The particle number was conserved by all processes and most operators.

Now we would like to introduce interactions between such fields. Unfortunately, interactions cannot be treated exactly.

We have to assume the strength of interactions to be sufficiently small. The well-understood free fields will dominate, and we insert interactions as small perturbations. This eventually leads us to Feynman diagrams to describe particle interactions order by order.

7.1 Interacting Lagrangians

One of the main reasons to consider QFT is its ability to deal with processes that do not conserve the number of particles.

Quantum fields are particle creation and annihilation operators: $\phi \to a, a^\dagger$. So far, we used them for two purposes:

- to build the multi-particle Fock space from a vacuum state, i.e. $a^\dagger \dots a^\dagger|0\rangle$;
- to write conserved charges as quadratic combinations of the fields which conspired to yield one creation and one annihilation operator $a^\dagger a$ and thus conserve the particle number.

Combining more than two fields typically yields a quantum operator which changes the particle number.

Time evolution of a quantum system is governed by its Hamiltonian, therefore it is natural to include such higher-order terms in it and consequently in the Lagrangian.

Scalar Interactions. For a real scalar field we could consider an interacting Lagrangian of the form

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{6}\mu\phi^3 - \tfrac{1}{24}\lambda\phi^4. \tag{7.1}$$

This model called the ϕ^4 theory.[1] It is perhaps the conceptually simplest interacting QFT model, but it leads to very non-trivial physics. The ϕ^4 term is also an interaction of the scalar Higgs field which is essential for the Higgs mechanism.

We might also add higher-order terms or terms involving derivatives such as

$$\phi^5, \quad \phi(\partial\phi)^2, \quad \phi^2(\partial\phi)^2, \quad (\partial\phi)^4, \quad \ldots. \tag{7.2}$$

Such terms are in principle allowable in QFT, but they have some undesirable features to be discussed later. We will, however, never add non-local terms of the type to the action

$$\int d^4x\, \phi(x)\phi(x+a), \qquad \int d^4x\, d^4y\, f(x,y)\phi(x)\phi(y). \tag{7.3}$$

These terms represent some unphysical action at a distance, we consider only local interactions which can be written using a local Lagrangian[2]

$$S = \int d^4x\, \mathcal{L}(x), \qquad \mathcal{L}(x) = \mathcal{L}[\phi(x), \partial\phi(x), \ldots]. \tag{7.4}$$

The equation of motion for the above Lagrangian reads

$$\partial^2\phi - m^2\phi - \tfrac{1}{2}\mu\phi^2 - \tfrac{1}{6}\lambda\phi^3 = 0. \tag{7.5}$$

It is a non-linear differential equation. Our usual strategy to deal with the differential equation of motion was to go to momentum space

$$\begin{aligned} 0 = & -p^2\phi(p) - m^2\phi(p) - \tfrac{1}{2}\mu \int \frac{d^4q}{(2\pi)^4}\, \phi(q)\phi(p-q) \\ & - \tfrac{1}{6}\lambda \int \frac{d^4q_1\, d^4q_2}{(2\pi)^8}\, \phi(q_1)\phi(q_2)\phi(p-q_1-q_2). \end{aligned} \tag{7.6}$$

Unfortunately, we obtain an integral equation instead of an algebraic equation. We cannot solve it in general, but for small $\mu \ll m$ and small $\lambda \ll 1$ we can try to find useful approximations.

Quantum Electrodynamics. Electrons and positrons carry an electrical charge $\mp q$ and their conserved current $J^\mu = -q\bar{\psi}\gamma^\mu\psi$ couples to the Maxwell equations. Putting together the Dirac and Maxwell Lagrangians we can simply add a source term for the electromagnetic potential $J^\mu A_\mu$

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + q\,\bar{\psi}\gamma^\mu\psi\, A_\mu. \tag{7.7}$$

[1] One often drops the term ϕ^3 and gains a discrete symmetry $\phi \mapsto -\phi$. A term ϕ^3 without a term ϕ^4 would lead to a potential unbounded from below.

[2] Non-local terms could be recovered as Taylor series involving derivatives of arbitrary order. It is therefore desirable to restrict the number of derivatives in $\mathcal{L}$.

This yields the desired inhomogeneous Maxwell equations, but also a modification of the Dirac equation

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -q\bar{\psi}\gamma^\nu\psi. \\ (\partial^\mu\gamma_\mu - m)\psi &= -q\gamma^\mu A_\mu\psi. \end{aligned} \tag{7.8}$$

The above model is called quantum electrodynamics (QED). It is a model that has been tested at a remarkable accuracy within its domain of validity, i.e. at low energies where the other elementary particles play no essential role. For instance, the electron anomalous magnetic dipole moment, also known as $g-2$, was predicted to more than 10 digits, and many of the leading digits are due to plain QED alone.

In the Standard Model, the above type of interaction between vectors and spinors is arguably the most important one because it couples matter in the form of spinor fields (leptons and quarks) to forces in the form of vector fields (photons, gluons and others).

Gauge Invariance. A crucial property of the electromagnetic potential is its gauge symmetry. In the quantisation procedure it eliminated an unphysical degree of freedom of the electromagnetic potential. We therefore want to preserve this symmetry in the presence of interactions.

The interaction term breaks the original gauge symmetry, but the latter can be restored by extending the symmetry to the Dirac field

$$\begin{aligned} A'_\mu(x) &= A_\mu(x) + \partial_\mu\alpha, \\ \psi'(x) &= \exp\bigl(iq\alpha(x)\bigr)\psi(x). \end{aligned} \tag{7.9}$$

Note that the latter transformation rule is just the global U(1) symmetry of the Dirac field which is responsible for conservation of the current J^μ. This global symmetry is enhanced to a local transformation parameter $\alpha(x)$. The derivative terms of $\alpha(x)$ are now compensated by the inhomogeneous gauge transformation of the potential A_μ.

There is a construction which makes the gauge invariance more manifest. Introduce the gauge covariant derivative

$$D_\mu = \partial_\mu - iqA_\mu. \tag{7.10}$$

Under gauge transformations this operator transforms homogeneously

$$\begin{aligned} D'_\mu &= \partial_\mu - iqA'_\mu = \partial_\mu - iqA_\mu - iq\partial_\mu\alpha \\ &= D_\mu + [D_\mu, -iq\alpha] \\ &= \exp(+iq\alpha)D_\mu\exp(-iq\alpha). \end{aligned} \tag{7.11}$$

In the QED Lagrangian written with a covariant derivative

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{7.12}$$

the factors of $\exp(\pm iq\alpha)$ trivially cancel between $\bar\psi$, D_μ and ψ. Moreover the electromagnetic field strength can be written as

$$F_{\mu\nu} \sim [D_\mu, D_\nu], \tag{7.13}$$

which makes manifest its invariance under gauge transformations.

Consequently, we can also couple the complex scalar field to the electromagnetic field via its kinetic term

$$\mathcal{L}_{\text{SQED}} = -(D^\mu\phi)^* D_\mu\phi - m^2|\phi|^2 - \tfrac{1}{4}\lambda|\phi|^4 - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{7.14}$$

This model is called scalar QED.

Further Interactions. Let us list some other simple interactions. We want to consider only those interactions which respect Lorentz symmetry. Curiously all of them appear in the Standard Model.

Two Dirac spinors can be multiplied to form a scalar combination. This can be multiplied by a scalar field

$$\bar\psi\psi\,\phi. \tag{7.15}$$

This term was originally proposed by Yukawa for the interaction between nucleons of spin $\frac{1}{2}$ and scalar pions. In the standard model such term couple the Higgs field to the leptons and quarks.

A similar term coupling a scalar and two Dirac fermions but with different parity properties is

$$\bar\psi i\gamma^5\psi\,\phi. \tag{7.16}$$

There is also an analog of the spinor-vector coupling with different parity properties

$$\bar\psi\gamma^5\gamma^\mu\psi\,A_\mu. \tag{7.17}$$

This axial vector coupling term is relevant to the weak nuclear interactions. Here, gauge invariance needs to extend to local chiral transformations of the spinors.

The above interactions for Dirac 4-spinors can be written in terms of more elementary chiral 2-spinor fields,

$$\chi^{\mathrm{T}}\sigma^2\chi\,\phi, \qquad \chi^\dagger\bar\sigma^\mu\chi\,A_\mu. \tag{7.18}$$

Note that the first interaction is complex, and therefore only some real projection can appear in the Lagrangian. This leads to two couplings, one for the real part and one for the imaginary part. The second term is perfectly real and requires a single real coupling constant.

Power Counting. We have encountered several types of interaction terms. These have a rather simple form with very few factors. Moreover, most of the simple terms have been observed directly or indirectly in nature. However, there

are many more local terms one could imagine, but which have not been observed. What distinguishes the above interactions?

To answer this question, consider the mass dimension. The action S must be a dimensionless quantity.[3] The action is the integral of the Lagrangian $S = \int d^4x\, \mathcal{L}$ and length counts as inverse mass, $dx \sim m^{-1}$, therefore the Lagrangian must have mass dimension 4,

$$\mathcal{L} \sim m^4. \tag{7.19}$$

The kinetic terms $(\partial\phi)^2$, $\bar{\psi}\partial\psi$ and F^2 where the derivative counts as a mass, $\partial \sim m$, determine the mass dimensions of the scalar, spinor and vector fields

$$\phi \sim A_\mu \sim m, \qquad \psi \sim m^{3/2}. \tag{7.20}$$

The mass dimension of the remaining terms is now fixed, e.g. for the mass terms

$$\phi^2 \sim m^2, \qquad \bar{\psi}\psi \sim m^3, \tag{7.21}$$

and for the simple interaction terms

$$\phi^3 \sim m^3, \qquad \phi^4 \sim \bar{\psi}\gamma^\mu\psi A_\mu \sim \bar{\psi}\psi\phi \sim m^4. \tag{7.22}$$

All of these terms have mass dimension at most 4. When they appear in the Lagrangian $\mathcal{L} \sim m^4$, their coupling constant must compensate for the missing mass dimension. The scalar and fermion mass terms therefore read $m^2\phi^2$, $m\bar{\psi}\psi$. Among the interaction terms, only ϕ^3 requires a dimensionful coupling $\mu \sim m$. All the other terms have mass dimension 4 and their coupling constants are plain numbers.

We can take the bound of mass dimension 4 [4] as an experimentally observed principle. There are good reasons to consider only terms of this type:

- Such interactions are reasonably simple.
- There are only finitely many such term, hence finitely many parameters for the model.
- All higher-dimensional terms require a coupling constant with negative mass dimension.
- Coupling constants with negative mass dimension lead to undesirable effects in the ultraviolet or short-distance regimes.
- Such theories are called non-renormalisable. Renormalisability will be considered later in QFT II.
- In the infrared or long-distance regime,[5] only the interactions of mass dimension up to 4 are relevant. The higher-dimensional terms have small effects are are mostly irrelevant.

[3] Quantities that appear in an exponent must be dimensionless numbers. The action carries the same units as Planck constant $\hbar$ which in natural units is a number $\hbar = 1$.

[4] More generally, the number of spacetime dimensions.

[5] The meaning of long-distance depends on the point of view. It can be astronomical units, everyday length scales, atomic scales or even less when interested in fundamental description of nature.

- The mass term for a vector field appears as an inverse power in the massive vector propagator. This also leads to a non-renormalisable model. To explain the mass of the W and Z vector bosons we rely on the Higgs mechanism.[6]

Symmetries. Unfortunately, it is generally hard to extract information from interacting QFT models. Usually we can only do certain approximations. Symmetries are powerful concepts in QFT because they can apply to interacting models as well:

- Free theories are somewhat trivial because there are infinitely many conservation laws. For example, the particle number is conserved, but it is related only to a non-local transformation.
- Only few of the conservation laws typically survive when interactions are added. Those are related to global symmetries preserved by the interactions.
- The conservation laws allow to make certain statements on the result of QFT observables even when computations are not feasible.
- Symmetries of the classical theory are not necessarily respected by the quantum theory. Such symmetries are called anomalous.
- Anomalies of local symmetries are typically bad because they spoil gauge redundancies which are required for consistency.

For example, consider ϕ^4 theory with a complex field given by the Lagrangian

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 |\phi|^2 - \tfrac{1}{4}\lambda |\phi|^4. \tag{7.23}$$

It is invariant under global multiplication by a complex phase $\phi \to e^{i\alpha}\phi$. This leads to the same conserved current as for the complex scalar

$$J^\mu = -i(\partial^\mu \phi^* \phi - \phi^* \partial^\mu \phi). \tag{7.24}$$

The associated current $Q = N_a - N_b$ is exactly conserved even in the presence of interactions. Conservation of the individual number operators N_a and N_b, however, is broken by interactions.

7.2 Interacting Field Operators

Consider an interacting field theory whose fields (and conjugate momenta) we will collectively denote by $\phi(x)$. More concretely, we can consider ϕ^4 theory.

We want to compute some correlation function, for example a correlator of two fields at different times t_1, t_2

$$F(x_2, x_1) = \langle 0|\phi(t_2, \vec{x}_2)\phi(t_1, \vec{x}_1)|0\rangle. \tag{7.25}$$

[6] Also known as the Englert–Brout–Higgs–Guralnik–Hagen–Kibble–Anderson–and–perhaps–also–'t–Hooft mechanism.

Interacting Field. As before, we can quantise the field ϕ on a time slice at some time t_0. This step is equivalent to a free field ϕ because the Poisson brackets are the same.

The full time dependence of ϕ is recovered by conjugating with the Hamiltonian

$$\phi(t, \vec{x}) = \exp(iH(t - t_0))\phi(\vec{x}) \exp(iH(t_0 - t)). \tag{7.26}$$

Supposing that the vacuum is time-invariant,[7] we can write the correlator as

$$F(x_2, x_1) = \langle 0|\phi(\vec{x}_2) \exp(-iH(t_2 - t_1))\phi(\vec{x}_1)|0\rangle. \tag{7.27}$$

Now everything is explicitly known except how to exponentiate H in practice. The latter is a hard problem.

Interaction Picture. We can do slightly better whenever the interactions are weak. In this case, the dominant contribution should come from the free Hamiltonian H_0. The quantisation of fields at a given time slice is the same. We can thus identify the fields at time t_0

$$\phi_0(\vec{x}) = \phi(\vec{x}). \tag{7.28}$$

Time evolution of the free field ϕ_0 is governed by the free Hamiltonian H_0

$$\phi_0(t, \vec{x}) = \exp(iH_0(t - t_0))\phi(\vec{x}) \exp(iH_0(t_0 - t)). \tag{7.29}$$

We know almost everything about this field. For weak interactions and small times $t \simeq t_0$, we expect the free field $\phi_0(t, \vec{x})$ to be a suitable approximation for the full field $\phi(t, \vec{x})$.

Comparing ϕ to ϕ_0 we can write

$$\phi(t, \vec{x}) = U(t, t_0)^{-1}\phi_0(t, \vec{x})U(t, t_0) \tag{7.30}$$

with the time evolution operator

$$U(t, t_0) = \exp\big(iH_0[\phi_0](t - t_0)\big) \exp\big(iH[\phi_0(t_0)](t_0 - t)\big). \tag{7.31}$$

For small interactions and small times, this operator is approximately the identity.

This is called the interaction picture, it is a mixture between the Schrödinger and the Heisenberg pictures

- In the Schrödinger picture, the field is defined on a constant time slice $\phi = \phi(t_0)$ and the operator $\exp(i(t - t_0)H)$ evolves states in time.
- In the Heisenberg picture, the field $\phi(t)$ carries the full time dependence, there is no need for a time evolution operator.
- In the interaction picture, the field $\phi_0(t)$ carries the time dependence of a free particle and the operator $U(t, t_0)$ evolves states in time.

[7] A constant energy of the vacuum can always be eliminated by subtracting it from H.

The correlator in question becomes

$$\langle 0|U(t_2,t_0)^{-1}\phi_0(t_2,\vec{x}_2)U(t_2,t_0)U(t_1,t_0)^{-1}\phi_0(t_1,\vec{x}_1)U(t_1,t_0)|0\rangle. \tag{7.32}$$

Note that products of time evolution operators can be joined in the obvious fashion, they form a group[8]

$$\begin{aligned}
& U(t_2,t_1)U(t_1,t_0) \\
=\ & \exp(iH_0[\phi_0](t_2-t_1))\exp(iH[\phi_0(t_1)](t_1-t_2)) \\
& \cdot\exp(iH_0[\phi_0](t_1-t_0))\exp(iH[\phi_0(t_0)](t_0-t_1)) \\
=\ & \exp(iH_0[\phi_0](t_2-t_1))\exp(iH_0[\phi_0](t_1-t_0)) \\
& \cdot\exp(iH[\phi_0(t_0)](t_1-t_2))\exp(iH[\phi_0(t_0)](t_0-t_1)) \\
=\ & \exp(iH_0[\phi_0](t_2-t_0))\exp(iH[\phi_0(t_0)](t_0-t_2)) \\
=\ & U(t_2,t_1).
\end{aligned} \tag{7.33}$$

We write this as

$$F(x_2,x_1)=\langle 0|U(t_0,t_2)\phi_0(x_2)U(t_2,t_1)\phi_0(x_1)U(t_1,t_0)|0\rangle. \tag{7.34}$$

Interacting Ground State. All the operators are expressed using the free field ϕ_0, but the state $|0\rangle$ is a state of the interacting theory and we do not know how to act on it.

Luckily we can express the interacting ground state $|0\rangle$ in terms of the vacuum $|0_0\rangle$ of the free theory with a trick: The free vacuum $|0_0\rangle$ should be some linear combination of the interacting ground state $|0\rangle$ and excited eigenstates $|n\rangle$ with definite energy $E_n > E_0$ [9]

$$|0_0\rangle = c_0|0\rangle + \sum_n c_n|n\rangle. \tag{7.35}$$

Letting this state evolve for some time T with the interacting Hamiltonian we obtain

$$\exp(-iHT)|0_0\rangle = \exp(-iE_0T)c_0|0\rangle + \sum_n c_n\exp(-iE_nT)|n\rangle. \tag{7.36}$$

All eigenstates oscillate with their respective frequencies. Suppose we give the time T some negative imaginary part with $(E_n-E_0)^{-1} \ll |\operatorname{Im} T| \ll |T|$. Then almost all eigenstates will get exponentially suppressed compared to the interacting ground state. The latter remains as the dominant contribution

$$\exp(-iHT)|0_0\rangle \approx \exp(-iE_0T)c_0|0\rangle. \tag{7.37}$$

Primarily this identification is a formal trick. In terms of physics, we let a system in some exited state $|0_0\rangle$ evolve for some long time and find it in its ground state

[8] It is crucial to note in terms of which fields the respective Hamiltonians are expressed: $H_0[\phi_0(t)]$ is independent of time, while $H[\phi_0(t)]$ depends on t due to the mismatch of fields.

[9] It is reasonable to assume $c_0 = \langle 0|0_0\rangle \neq 0$ when interactions are sufficiently small.

$|0\rangle$. This is reasonable if we assume the system to be open or damped in some way. All real world systems, at least those we can expect to observe, are finite and open; the effect of $i\epsilon$ is to implement this assumption into our calculations.

We can thus express the interacting vacuum at some time t_0 as the evolution of the free vacuum at time $-T$

$$|0\rangle \simeq U(t_0, -T)|0_0\rangle, \tag{7.38}$$

where we did not pay attention to normalisation. Analogously,

$$\langle 0| \simeq \langle 0_0|U(+T, t_0). \tag{7.39}$$

Our final result for the correlation function $F(x_2, x_1)$:

$$\lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0_0|U(T, t_2)\phi_0(x_2)U(t_2, t_1)\phi_0(x_1)U(t_1, -T)|0_0\rangle}{\langle 0_0|U(T, -T)|0_0\rangle}\,. \tag{7.40}$$

The denominator implements the desired normalisation $\langle 0|0\rangle = 1$.

Interacting Correlators. In conclusion, the recipe for determining some correlation function in the interacting theory is the following

$$\langle 0|X|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0_0|U(T, t_0)XU(t_0, -T)|0_0\rangle}{\langle 0_0|U(T, -T)|0_0\rangle}\,. \tag{7.41}$$

where all the interacting quantum operators in X are replaced by free fields evolved from time t_0 to the desired time slice

$$\phi(t, \vec{x}) \to U(t_0, t)\phi_0(t, \vec{x})U(t, t_0). \tag{7.42}$$

Effectively two consecutive time evolution operators can always be combined into one

$$U(t_2, t_0)U(t_0, t_1) = U(t_2, t_1). \tag{7.43}$$

7.3 Perturbation Theory

We still cannot evaluate the time evolution operator $U(t, t_0)$, but at least we know that it is close to the identity when interactions are sufficiently small

$$U(t, t_0) \approx 1. \tag{7.44}$$

This approximation is too crude, it is equivalent to computing the correlator in the free theory, and we gain nothing.

Schrödinger Equation. To improve the approximation, consider the time derivative of $U(t, t_0)$

$$\begin{aligned} i\partial_t U(t, t_0) =& \exp(iH_0(t - t_0))\big(H[\phi_0(t_0)] - H_0\big) \\ & \cdot \exp(-iH_0(t - t_0))U(t, t_0). \\ =& \big(H[\phi_0(t)] - H_0\big)U(t, t_0). \end{aligned} \tag{7.45}$$

We see that the time evolution operator is determined by a differential equation and a trivial initial value condition

$$i\partial_t U(t,t_0) = H_{\rm int}(t)U(t,t_0), \qquad U(t_0,t_0) = 1. \tag{7.46}$$

This is a Schrödinger equation, and its Hamiltonian is the so-called interaction Hamiltonian

$$H_{\rm int}(t) := H[\phi_0(t)] - H_0[\phi_0]. \tag{7.47}$$

This Hamiltonian is time-dependent, therefore the solution cannot be as simple as $\exp(-i(t-t_0)H_{\rm int})$.[10] For weak interactions, one can use the Dyson series to solve the equation perturbatively.

Dyson Series. The interaction Hamiltonian is the quantity which we should assume to be small. It appears in the Schrödinger equation, so at first order we can use the above approximation for U

$$i\partial_t U(t,t_0) = H_{\rm int}(t)U(t,t_0) \approx H_{\rm int}(t). \tag{7.48}$$

Integrating with proper initial value this yields

$$U(t,t_0) \approx 1 - i\int_{t_0}^{t} dt_1 H_{\rm int}(t_1). \tag{7.49}$$

This is certainly better than before, it involves interactions at first order. Nevertheless we can do better.

To go further systematically, write the differential equation in integral form

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt_1\, H_{\rm int}(t_1)U(t_1,t_0). \tag{7.50}$$

Substitute the above solution yields a better solution

$$\begin{aligned} U(t,t_0) \approx 1 &- i\int_{t_0}^{t} dt_1\, H_{\rm int}(t_1) \\ &- \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_{\rm int}(t_1)\, H_{\rm int}(t_2). \end{aligned} \tag{7.51}$$

Now use the new solution instead

$$\begin{aligned} U(t,t_0) \approx 1 &- i\int_{t_0}^{t} dt_1\, H_{\rm int}(t_1) \\ &- \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_{\rm int}(t_1)H_{\rm int}(t_2) \\ &+ i\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3\, H_{\rm int}(t_1)\, H_{\rm int}(t_2)\, H_{\rm int}(t_3). \end{aligned} \tag{7.52}$$

And so on.

The picture should be clear, we could go to arbitrarily high orders. More importantly, everything is expressed in terms of free fields ϕ_0 and the interaction Hamiltonian $H_{\rm int}[\phi_0]$.

[10] $H_{\rm int}$ is time-dependent because its time evolution is governed by H_0 with which it does not commute in general.

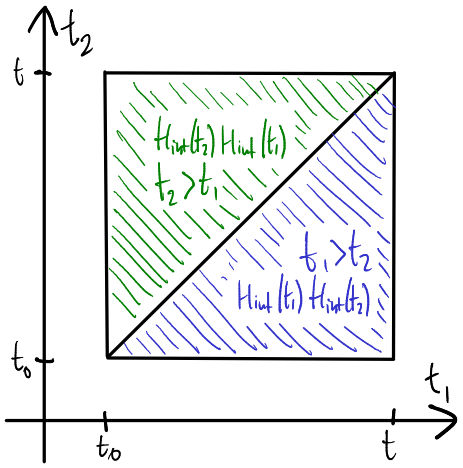
Time-Ordered Exponential. The multiple integral with a nested sequence of boundaries is hard to handle. We can improve the situation. Consider the quadratic term:

$$-\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_{\rm int}(t_1)\, H_{\rm int}(t_2). \tag{7.53}$$

We can also write it as

$$-\int_{t_0}^{t} dt_1 \int_{t_1}^{t_2} dt_2\, H_{\rm int}(t_2)\, H_{\rm int}(t_1). \tag{7.54}$$

The integration region assumes $t_1 \geq t_2$ in the first integral and $t_2 \geq t_1$ in the second integral. Importantly, in both integrands the operator $H_{\rm int}(t_k)$ with larger t_k is to the right of the operator $H_{\rm int}(t_j)$ with smaller t_j.



(7.55)

We introduce a time ordering symbol T which puts the affected operators in an order with time decreasing from left to right, e.g.[11]

$$\mathrm{T}\big(X(t_1)Y(t_2)\big) := \begin{cases} X(t_1)Y(t_2) & \text{for } t_1 > t_2, \\ Y(t_2)X(t_1) & \text{for } t_1 < t_2, \end{cases} \tag{7.56}$$

and similarly for multiple operators. This allows to write the integrand of both above integrals as $\mathrm{T}(H_{\rm int}(t_1)\, H_{\rm int}(t_2))$. We can thus write the integral as the average of the two equivalent representations where the integration regions combine to a square

$$-\frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, \mathrm{T}\big(H_{\rm int}(t_1)\, H_{\rm int}(t_2)\big). \tag{7.57}$$

Even better, we can write this as the time-ordered square of a single integral

$$-\frac{1}{2}\mathrm{T}\left(\int_{t_0}^{t} dt'\, H_{\rm int}(t')\right)^2 \tag{7.58}$$

[11] For fermionic operators X, Y one would insert suitable signs for flipping the order.

As all terms of the perturbative expansion of $U(t, t_0)$ are naturally in time ordering, the above construction generalises straight-forwardly to the n-th order term

$$\frac{1}{n!}\mathrm{T}\left(-i\int_{t_0}^{t} dt'\, H_{\mathrm{int}}(t')\right)^n . \tag{7.59}$$

Here the integration region is a hypercube in n dimensions. It contains $n!$ simplices,[12] which form the integration regions for the terms in the Dyson series.

Summing up all terms yields the time-ordered exponential

$$U(t_2, t_1) = \mathrm{T}\exp\left(iS_{\mathrm{int}}(t_1, t_2)\right). \tag{7.60}$$

where we introduced the interaction action S_{int} between times t_1 and t_2

$$S_{\mathrm{int}}(t_1, t_2) := -\int_{t_1}^{t_2} dt'\, H_{\mathrm{int}}(t'). \tag{7.61}$$

The time-ordered exponential represents both the formal solution to the above Schrödinger equation for $U(t_2, t_1)$ as well as a concrete perturbative prescription to evaluate it.

[12] A hypercube is the generalisation of a cube to n dimensions, a simplex is the generalisation of a triangle.