6 Free Vector Field

Next we want to find a formulation for vector fields. This includes the important case of the electromagnetic field with its photon excitations as massless relativistic particles of helicity 1. This field will be the foundation for a QFT treatment of electrodynamics called quantum electrodynamics (QED). Here we will encounter a new type of symmetry which will turn out to be extremely powerful but at the price of new complications.

6.1 Classical Electrodynamics

We start by recalling electrodynamics which is the first classical field theory most of us have encountered in theoretical physics.

Maxwell Equations. The electromagnetic field consists of the electric field $\vec{E}(t,\vec{x})$ and the magnetic field $\vec{B}(t,\vec{x})$. These fields satisfy the four Maxwell equations (with $\varepsilon_0 = \mu_0 = c = 1$)

$$\begin{aligned}
0 &= \operatorname{div}\vec{B} := \vec{\partial}\cdot\vec{B} = \partial_k B_k,\\
0 &= \operatorname{rot}\vec{E} + \dot{\vec{B}} := \vec{\partial}\times\vec{E} + \dot{\vec{B}} = \varepsilon_{ijk}\partial_j E_k + \dot{B}_i,\\
\rho &= \operatorname{div}\vec{E} = \vec{\partial}\cdot\vec{E} = \partial_k E_k,\\
\vec{\jmath} &= \operatorname{rot}\vec{B} - \dot{\vec{E}} = \vec{\partial}\times\vec{B} - \dot{\vec{E}} = \varepsilon_{ijk}\partial_j B_k - \dot{E}_i.
\end{aligned} \tag{6.1}$$

The fields ρ and $\vec{\jmath}$ are the electrical charge and current densities.

The solutions to the Maxwell equations without sources are waves propagating with the speed of light. The Maxwell equations were the first relativistic wave equations that were found. Eventually their consideration led to the discovery of special relativity.

Relativistic Formulation. Lorentz invariance of the Maxwell equations is not evident in their usual form. Let us transform them to a relativistic form.

The first step consists in converting B_i to an anti-symmetric tensor of rank 2

$$B_i = -\tfrac{1}{2}\varepsilon_{ijk}F_{jk}, \qquad F = \begin{pmatrix} 0 & -B_z & +B_y \\ +B_z & 0 & -B_x \\ -B_y & +B_x & 0 \end{pmatrix}. \tag{6.2}$$

Then the Maxwell equations read

$$\begin{aligned}
0 &= -\varepsilon^{ijk}\partial_k F_{ij}, & \rho &= \partial_k E_k,\\
0 &= \varepsilon^{ijk}(2\partial_j E_k - \dot{F}_{jk}), & \vec{\jmath} &= \partial_j F_{ji} - \dot{E}_i.
\end{aligned} \tag{6.3}$$

These equations are the $1+3$ components of two 4-vectors which can be seen by setting

$$E_k = F_{0k} = -F_{k0}, \qquad J^\mu = (\rho, \vec{\jmath}). \tag{6.4}$$

Now the Maxwell equations simply read

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \qquad \partial_\nu F^{\nu\mu} = J^\mu. \tag{6.5}$$

Electromagnetic Potential. For QFT purposes we need to write a Lagrangian from which the Maxwell equations follow. This is however not possible using $F_{\mu\nu}$ as the fundamental degrees of freedom. A Lagrangian can be constructed by the help of the electromagnetic vector potential A_μ. This is not just a technical tool, but it will be necessary to couple the field to charged matter. This fact can be observed in the Aharonov–Bohm effect, where a quantum particle feels the presence of a non-trivial electromagnetic potential A, although it is confined to a region of spacetime where the field strength vanishes $F = 0$.

The first (homogeneous) equation is an integrability condition for the field $F_{\mu\nu}$. It implies that it can be integrated consistently to an electromagnetic potential A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.6}$$

With this parametrisation of F the homogeneous equation is automatically satisfied.

The electromagnetic potential is not uniquely defined by the electromagnetic fields F. For any solution A, we can add the derivative of a scalar field

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x). \tag{6.7}$$

The extra term cancels out when anti-symmetrising the two indices in $F_{\mu\nu}$ and hence

$$F'_{\mu\nu}(x) = F_{\mu\nu}(x). \tag{6.8}$$

This freedom in defining A_μ is called a gauge symmetry or gauge redundancy. It is called a local symmetry because the transformation can be chosen independently for every point of spacetime. Gauge symmetry will turn out very important in quantising the vector field.

Lagrangian. A Lagrangian for the electromagnetic fields can now be formulated in terms of the potential A_μ

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}[A]F_{\mu\nu}[A] = \tfrac{1}{2}\vec{E}[A]^2 - \tfrac{1}{2}\vec{B}[A]^2. \tag{6.9}$$

Here and in the following, $F_{\mu\nu}[A]$ is not considered a fundamental field, but merely represents the combination

$$F_{\mu\nu}[A] = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.10}$$

The equation of motion yields the second (inhomogeneous) Maxwell equation (here with a trivial source term)

$$\partial_\nu F^{\nu\mu} = 0. \tag{6.11}$$

The first (homogeneous) Maxwell equation is already implied by the definition of F in terms of A.

Due to Poincaré symmetry we can also derive an energy momentum tensor $T_{\mu\nu}$. It takes the form[1]

$$T_{\mu\nu} = -F_\mu{}^\rho F_{\nu\rho} + \tfrac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}. \tag{6.12}$$

6.2 Gauge Fixing

Hamiltonian Framework. Towards quantisation we should proceed to the Hamiltonian framework. The canonical momentum $\mathit{\Pi}$ conjugate to the field A reads

$$\mathit{\Pi}_\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}^\mu} = F_{0\mu}. \tag{6.13}$$

Here a complication arises because the component $\mathit{\Pi}_0$ is strictly zero and the field A_0 has no conjugate momentum. The non-zero components are the electrical field $F_{0k} = E_k$. Moreover the equations of motion imply $\partial_k\mathit{\Pi}_k = 0$ which is an equation without time derivative.

The missing of the momentum $\mathit{\Pi}_0$ and the spatial differential equation for $\mathit{\Pi}_k$ are two constraints which the momenta will have to satisfy, even in the initial condition. These are so-called constraints. For the massless vector field they are related to gauge redundancy of A. Although A_μ has four components, one of them can be chosen arbitrarily using gauge symmetry. Effectively A_μ has only three physically relevant components, which is matched by only three conjugate momenta.

Coulomb Gauge. A simple ansatz to resolve the problem of $\mathit{\Pi}_0 = 0$ is to demand that

$$A_0 = 0. \tag{6.14}$$

This can always be achieved by a suitable gauge transformation.

This choice does not completely eliminate all gauge freedom for A_k, a time-independent gauge redundancy $\alpha(\vec{x})$ remains. It can be eliminated by the demanding

$$\partial_k A_k = 0 \tag{6.15}$$

which is called the Coulomb gauge (fixing).

Now $\mathit{\Pi}_k = F_{0k} = E_k = \dot{A}_k$ and for the Hamiltonian we obtain

$$H = \int d^3x\, \tfrac{1}{2}(\vec{E}^{\,2} + \vec{B}^{\,2}), \tag{6.16}$$

[1]The naive derivation from the Lagrangian yields $T_{\mu\nu} = -F_{\mu\rho}\partial_\nu A^\rho - \frac{1}{4}\mathcal{L}$ which is neither symmetric nor gauge invariant. It is repaired by adding the term $\partial^\rho(F_{\mu\rho}A_\nu)$.

which indeed represents the energy of the electromagnetic field.

With the Coulomb gauge, we can now quantise the electromagnetic field. The gauge is however not always convenient, since it specialises the time direction and therefore breaks relativistic invariance. For instance, it leads to instantaneous contributions to field correlators, which may feel odd. In physical observables, eventually such instantaneous or causality-violating contributions will always cancel.

Lorenz Gauges. A more general class gauge fixings are the Lorenz gauges

$$\partial^\mu A_\mu = 0. \tag{6.17}$$

Again, they do not completely fix the gauge freedom since any gauge transformation with $\partial^2\alpha = 0$ will preserve the Lorenz gauge condition. For example, one may furthermore demand $A_0 = 0$ to recover the Coulomb gauge.

It is convenient to implement the Lorenz gauge by adding a gauge fixing term $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\xi(\partial{\cdot}A)^2$ to the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{ED}} + \mathcal{L}_{\text{gf}} \simeq -\tfrac{1}{2}\partial^\mu A^\nu\,\partial_\mu A_\nu + \tfrac{1}{2}(1-\xi)\partial^\mu A_\mu \partial^\nu A_\nu. \tag{6.18}$$

The equations of motion now read

$$\partial^2 A_\mu - (1-\xi)\partial_\mu\partial^\nu A_\nu = 0 \tag{6.19}$$

Let us show that the equations require $\partial^2 A_\mu = 0$ and $\partial \cdot A = 0$. We solve the equation in momentum space

$$p^2 A_\mu - (1-\xi)p_\mu(p{\cdot}A) = 0 \tag{6.20}$$

We first multiply the equation by p^μ to get

$$\xi p^2(p{\cdot}A) = 0. \tag{6.21}$$

Unless $\xi = 0$, this equation implies that $p^2 = 0$ or $p{\cdot}A = 0$. Using this result in the original equation shows that both $p^2 = 0$ and $p{\cdot}A = 0$ must hold unless $\xi = 1$. [2] For $\xi = 1$, the equation only requires $p^2 = 0$.

The canonical momenta now read

$$\mathit{\Pi}_\mu = \dot{A}_\mu + \delta^0_\mu(1-\xi)(\partial^\nu A_\nu), \tag{6.22}$$

which can be solved for all $\dot{A}_\mu$ unless $\xi = 0$. We can now define canonical Poisson brackets

$$\{A_\mu(t,\vec{x}), \mathit{\Pi}^\nu(t,\vec{y})\} = \delta^\nu_\mu \delta^3(\vec{x}-\vec{y}). \tag{6.23}$$

[2] This statement might not be accurate in a distributional sense! For example, the function $A_\mu = x_\mu$ satisfies the equation of motion, but yields $\partial{\cdot}A = -4 \neq 0$. In momentum space this function is a distribution.

There appears to be a catch: The gauge fixing condition $\partial{\cdot}A = 0$ constrains the conjugate momentum $\mathit{\Pi}_0$

$$\mathit{\Pi}_0 = \partial_k A_k - \xi(\partial^\nu A_\nu). \tag{6.24}$$

to be a function of the A_k alone. Substituting this into the Poisson brackets yields the identity

$$\{A_0, \mathit{\Pi}_0\} = \{A_0, \partial_k A_k\} - \xi\{A_0, \partial^\nu A_\nu\} = -\xi\{A_0, \partial^\nu A_\nu\}, \tag{6.25}$$

where we used that the A's and their spatial derivatives commute. Here the term on the left hand side is non-trivial whereas the term on the right hand side should be trivial due to $\partial{\cdot}A = 0$.[3]

Feynman Gauge. To avoid these problems and also to simplify the subsequent analysis, we shall set $\xi = 1$. This so-called Feynman gauge[4] has a simple Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu A^\nu\, \partial_\mu A_\nu \tag{6.26}$$

with simple equations of motion

$$\partial^2 A_\mu = 0. \tag{6.27}$$

Effectively, it describes 4 massless scalar fields A_μ with the peculiarity that the sign of the kinetic term for A_0 is wrong. With the canonical momenta $\mathit{\Pi}_\mu = \dot{A}_\mu$ the Poisson brackets read

$$\{A_\mu(\vec{x}), \mathit{\Pi}_\nu(\vec{y})\} = \eta_{\mu\nu}\delta^3(\vec{x} - \vec{y}). \tag{6.28}$$

where again the relation for A_0 has the opposite sign. Likewise all correlation functions and propagators equal their scalar counterparts times $\eta_{\mu\nu}$.

As such, the model described by the above simple Lagrangian is not electrodynamics. Only when taking into account the constraint $\partial{\cdot}A = 0$ it becomes electrodynamics. Moreover, the constraint will be crucial in making the QFT model physically meaningful. Nevertheless we have to be careful in implementing the constraint since it is inconsistent with the Poisson brackets.

Light Cone Gauge. The above Lorenz gauges do not eliminate all unphysical degrees of freedom, which introduce some complications later. There are other useful gauges which avoid these problems, but trade them in for others. A prominent example is the light cone gauge which eliminates a light-like component $A_- = A_0 - A_3 = 0$ of the gauge potential A_μ. The equations of motion then allow to solve for a non-collinear like-like component of $A_+ = A_0 + A_3$. The remaining two degrees of freedom of A_μ then represent the two helicity modes of the electromagnetic field. Let us nevertheless continue in the Feynman gauge.

---

[3] However, note that $\partial{\cdot}A = 0$ need not hold in a strict sense, i.e. the above Poisson brackets are self-consistent and can be quantised.

[4] Actually, it is a gauge fixing term rather than a gauge.

6.3 Particle States

Next we quantise the model and discuss its particle states. The construction of Fock space is the same as for a set of four massless scalar fields, but we need to implement the gauge-fixing constraint.

Quantisation. We quantise the vector field $A_\mu(x)$ in Feynman gauge analogously to four independent scalar fields where merely one of the kinetic term has the opposite sign. This leads to the equal-time commutation relations

$$[A_\mu(t,\vec{x}), \dot{A}_\nu(t,\vec{y})] = i\eta_{\mu\nu}\delta^3(\vec{x}-\vec{y}). \tag{6.29}$$

We then solve the equation of motion $\partial^2 A_\mu = 0$ in momentum space

$$A_\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2e(\vec{p})} \left(e^{-ip\cdot x} a_\mu(\vec{p}) + e^{ip\cdot x} a^\dagger_\mu(\vec{p})\right) \tag{6.30}$$

and translate the above field commutators to commutators for creation and annihilation operators

$$[a_\mu(\vec{p}), a^\dagger_\nu(\vec{q})] = \eta_{\mu\nu}\, 2e(\vec{p})\, (2\pi)^3\, \delta^3(\vec{p}-\vec{q}). \tag{6.31}$$

Fock Space. We define the vacuum state $|0\rangle$ to be annihilated by all $a_\mu(\vec{p})$

$$a_\mu(\vec{p})|0\rangle = 0. \tag{6.32}$$

As before, multi-particle states are constructed by acting with the creation operators $a^\dagger_\mu(\vec{p})$ on the vacuum $|0\rangle$.

There are two problems with this naive Fock space. The first is that there ought to be only two states (with helicity $h = \pm 1$) for each momentum. Here we have introduced four states. The other problem is that one of these states has a negative norm: To see this we prepare a wave packet for $a^\dagger_0$

$$|f\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2e(\vec{p})}\, f(\vec{p}) a^\dagger_0(\vec{p})|0\rangle. \tag{6.33}$$

The norm of this state is negative definite

$$\langle f|f\rangle = -\int \frac{d^3\vec{p}}{(2\pi)^3\, 2e(\vec{p})}\, |f(\vec{p})|^2 < 0. \tag{6.34}$$

A negative-norm state violates the probabilistic interpretation of QFT, hence it must be avoided at all means.

Physical States. The above problems are eventually resolved by implementing the gauge-fixing constraint $\partial{\cdot}A = 0$ which we have not yet considered. This is not straight-forward:

- The commutation relations prevent us from implementing it at an operatorial level.
- We cannot implement it directly on states: E.g. requiring the vacuum $|0\rangle$ to be physical means setting $p{\cdot}a^\dagger(\vec{p})|0\rangle = 0$ which is inconsistent with the commutation relations.
- The weakest implementation is to demand that the expectation value of $\partial{\cdot}A$ vanishes for all physical states. This is the Gupta–Bleuler formalism.

For two physical states $|\Psi\rangle, |\Phi\rangle$ we thus demand

$$\langle\Phi|\partial{\cdot}A|\Psi\rangle = 0. \tag{6.35}$$

This is achieved by demanding

$$p{\cdot}a(\vec{p})|\Psi\rangle = 0 \tag{6.36}$$

for any physical state. An adjoint physical state then obeys $\langle\Phi|p{\cdot}a^\dagger(\vec{p}) = 0$.

- Both conditions together ensure that $\langle\Phi|\partial{\cdot}A|\Psi\rangle = 0$.
- Moreover, the vacuum is physical by construction.

We conclude that Fock space is too large. The space of physical states $|\Psi\rangle$ is a subspace of Fock space such that for all $\vec{p}$

$$p{\cdot}a(\vec{p})|\Psi\rangle = 0. \tag{6.37}$$

Nevertheless we cannot completely abandon the larger Fock space in favour of the smaller space of physical states. For instance, the action of $A_\mu(x)$ cannot be confined to the physical subspace since it does not commute with the operator $p{\cdot}a(\vec{p})$.

Evidently, the negative-norm state $|f\rangle$ discussed above is not physical since

$$p{\cdot}a(\vec{p})|f\rangle = f(\vec{p})e(\vec{p})|0\rangle. \tag{6.38}$$

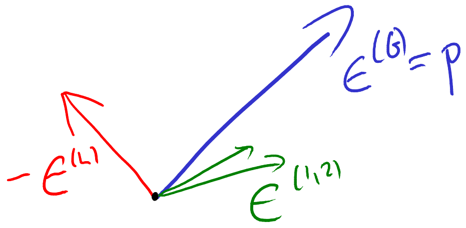
It is an element of Fock space, but not of its physical subspace.

Basis of Polarisation Vectors. To investigate the space of physical states further, we introduce a convenient basis for polarisation vectors $\epsilon^{(\alpha)}_\mu(\vec{p})$ of the vector field $a_\mu(\vec{p})$ and $a^\dagger_\mu(\vec{p})$ on the light cone $p^2 = 0$.[5]

We denote the four polarisations α by G for gauge, L for longitudinal and $1, 2$ for the two transverse directions.

- We first define $\epsilon^{(\mathrm{G})}$ as a light-like vector in the direction of p, e.g. $\epsilon^{(\mathrm{G})} = p$.
- We construct another light-like vector $\epsilon^{(\mathrm{L})}$ which has unit scalar product with $\epsilon^{(\mathrm{G})}$, i.e. $\epsilon^{(\mathrm{L})}{\cdot}\epsilon^{(\mathrm{G})} = 1$.
- We then construct two orthonormal space-like vectors $\epsilon^{(1,2)}$ which are also orthogonal to $\epsilon^{(\mathrm{G})}$ and $\epsilon^{(\mathrm{L})}$.

[5] The polarisation vectors are similar to the spinors $u(\vec{p})$ and $v(\vec{p})$ for the Dirac equation.



(6.39)

For example, suppose the light-like momentum is given by

$$p_\mu = (e, 0, 0, e). \tag{6.40}$$

Then we can define the following 4 vectors [6]

$$\begin{aligned}
\epsilon^{(\mathrm{G})}_\mu(\vec{p}) &= (e, 0, 0, e),\\
\epsilon^{(\mathrm{L})}_\mu(\vec{p}) &= (-1/2e, 0, 0, 1/2e),\\
\epsilon^{(1)}_\mu(\vec{p}) &= (0, 1, 0, 0),\\
\epsilon^{(2)}_\mu(\vec{p}) &= (0, 0, 1, 0).
\end{aligned} \tag{6.41}$$

These four polarisations define a complete basis for vector space. We can thus decompose the creation and annihilation operators as follows

$$a^{(\dagger)}_\mu = \epsilon^{(\mathrm{G})}_\mu a^{(\dagger)}_{(\mathrm{L})} + \epsilon^{(\mathrm{L})}_\mu a^{(\dagger)}_{(\mathrm{G})} + \epsilon^{(1)}_\mu a^{(\dagger)}_{(1)} + \epsilon^{(2)}_\mu a^{(\dagger)}_{(2)}. \tag{6.42}$$

Likewise we can write the commutation relations[7]

$$\begin{aligned}
&[a_{(\mathrm{L})}(\vec{p}), a^\dagger_{(\mathrm{G})}(\vec{q})] = [a_{(1)}(\vec{p}), a^\dagger_{(1)}(\vec{q})] =\\
&[a_{(\mathrm{G})}(\vec{p}), a^\dagger_{(\mathrm{L})}(\vec{q})] = [a_{(2)}(\vec{p}), a^\dagger_{(2)}(\vec{q})] = 2e(\vec{p})\,(2\pi)^3\delta^3(\vec{p}-\vec{q}).
\end{aligned} \tag{6.43}$$

By construction we know that

$$p{\cdot}a(\vec{p}) = \epsilon^{(\mathrm{G})}{\cdot}a(\vec{p}) = a_{(\mathrm{G})}(\vec{p}). \tag{6.44}$$

hence the physical state condition in this basis reads

$$a_{(\mathrm{G})}(\vec{p})|\Psi\rangle = 0. \tag{6.45}$$

The physical state condition together with the commutation relations implies that a physical state cannot have any longitudinal excitations $a^\dagger_{(\mathrm{L})}(\vec{p})$. It must be of the form[8]

$$|\Psi\rangle = a^\dagger_{(\mathrm{G})}\cdots a^\dagger_{(\mathrm{G})}a^\dagger_{(1,2)}\cdots a^\dagger_{(1,2)}|0\rangle. \tag{6.46}$$

Since negative norm states can originate exclusively from the commutators $[a_{(\mathrm{L})}, a^\dagger_{(\mathrm{G})}]$ and $[a_{(\mathrm{G})}, a^\dagger_{(\mathrm{L})}]$, and since the $a^\dagger_{(\mathrm{L})}$'s are absent, the norm of any such state is positive semi-definite

$$\langle\Psi|\Psi\rangle \geq 0. \tag{6.47}$$

The modes $a^\dagger_{(1,2)}$ have a positive norm while $a^\dagger_{(\mathrm{G})}$ is null.

---

[6] There is a lot of arbitrariness in defining the polarisation vectors for each momentum p, but it does not matter.

[7] The crossing between L and G is due to the construction of the basis using two light-like directions.

[8] Note that $a_{(\mathrm{G})}$ commutes with $a^\dagger_{(\mathrm{G})}$ and $a^\dagger_{(1,2)}$ but not with $a^\dagger_{(\mathrm{L})}$.

Null States. Consider a physical state $|\mathit{\Psi}\rangle$ which contains an excitation of type $a^\dagger_{(\mathrm{G})}$, i.e. a state which can be written as

$$|\mathit{\Psi}\rangle = a^\dagger_{(\mathrm{G})}(\vec{p})|\mathit{\Omega}\rangle \tag{6.48}$$

with some other physical state $|\mathit{\Omega}\rangle$. This state has zero norm by the physical state condition

$$\langle\mathit{\Psi}|\mathit{\Psi}\rangle = \langle\mathit{\Omega}|a_{(\mathrm{G})}(\vec{p})a^\dagger_{(\mathrm{G})}(\vec{p})|\mathit{\Omega}\rangle = \langle\mathit{\Omega}|a^\dagger_{(\mathrm{G})}(\vec{p})a_{(\mathrm{G})}(\vec{p})|\mathit{\Omega}\rangle = 0. \tag{6.49}$$

Null states are not normalisable and therefore have to be interpreted appropriately. Typically null states are irrelevant because QM is a probabilistic framework. Something that takes place with probability zero does not happen. Nevertheless, some consistency requirements have to be fulfilled:

By the same argument as above, we can show that a null state

$$|\mathit{\Psi}\rangle = a^\dagger_{(\mathrm{G})}(\vec{p})|\mathit{\Omega}\rangle \tag{6.50}$$

actually has vanishing scalar products with any physical state $|\mathit{\Phi}\rangle$ due to the physicality condition of the latter

$$\langle\mathit{\Phi}|\mathit{\Psi}\rangle = \langle\mathit{\Phi}|a^\dagger_{(\mathrm{G})}(\vec{p})|\mathit{\Omega}\rangle = 0. \tag{6.51}$$

In particular, this implies that the sum $|\mathit{\Psi}'\rangle$ of a physical state $|\mathit{\Psi}\rangle$ and some null state

$$|\mathit{\Psi}'\rangle = |\mathit{\Psi}\rangle + a^\dagger_{(\mathrm{G})}(\vec{p})|\mathit{\Phi}\rangle \tag{6.52}$$

behaves just like the original physical state $|\mathit{\Psi}\rangle$ in scalar products

$$\langle\mathit{\Phi}|\mathit{\Psi}'\rangle = \langle\mathit{\Phi}|\mathit{\Psi}\rangle + \langle\mathit{\Phi}|a^\dagger_{(\mathrm{G})}(\vec{p})|\mathit{\Omega}\rangle = \langle\mathit{\Phi}|\mathit{\Psi}\rangle. \tag{6.53}$$

We should thus impose an equivalence relation on the physical Fock space

$$|\mathit{\Psi}\rangle \simeq |\mathit{\Psi}'\rangle = |\mathit{\Psi}\rangle + a^\dagger_{\mathrm{G}}(\vec{p})|\mathit{\Omega}\rangle. \tag{6.54}$$

Any two states which differ by a state which is in the image of some $a^\dagger_{(\mathrm{G})}$ are physically equivalent. In other words, physical states of the gauge field are not described by particular vectors but by equivalence classes of vectors.

We may use states which have no contribution of $a_{(\mathrm{G})}$ as reference states of the equivalence classes[9]

$$|\mathit{\Psi}\rangle = a^\dagger_{(1,2)} \cdots a^\dagger_{(1,2)}|0\rangle. \tag{6.55}$$

These representatives show that we have two states for each momentum $\vec{p}$. It matches nicely with the massless UIR's of the Poincaré group with positive and negative helicity $h = \pm 1$. The particle excitations of the electromagnetic field are the photons.

[9]Although this appears to be a useful choice at first sight, it is not at all unique. By a change of basis for the polarisation vectors at any given $\vec{p}$ we can add any amount of $a^\dagger_{(\mathrm{G})}$ to $a^\dagger_{(1,2)}$. The new states are certainly in the same equivalence class, but they are different representatives.

Gauge Transformations. However, inserting some gauge potentials $A_\mu(x)$ into the scalar product may actually lead to some dependence on null states. Let us therefore compute

$$\begin{aligned}\langle\Phi|A_\mu(x)|\Psi'\rangle &= \langle\Phi|A_\mu(x)|\Psi\rangle + \langle\Phi|A_\mu(x)a^\dagger_{(\mathrm{G})}(\vec{p})|\Omega\rangle\\ &= \langle\Phi|A_\mu(x)|\Psi\rangle + \langle\Phi|[A_\mu(x), a^\dagger_{(\mathrm{G})}(\vec{p})]|\Omega\rangle\end{aligned} \tag{6.56}$$

We write $a^\dagger_{(\mathrm{G})}(\vec{p}) = p{\cdot}a^\dagger(\vec{p})$ and the commutator evaluates to

$$[A_\mu(x), a^\dagger_{(\mathrm{G})}(\vec{p})] = p_\mu e^{-ip\cdot x} = i\partial_\mu e^{-ip\cdot x} \tag{6.57}$$

The expectation value of $A_\mu(x)$ thus changes effectively by a derivative term

$$A_\mu(x) \mapsto A_\mu(x) + \frac{\langle\Phi|\Omega\rangle}{\langle\Phi|\Psi\rangle}\, i\partial_\mu e^{-ip\cdot x}. \tag{6.58}$$

This is just a gauge transformation of the potential $A_\mu(x)$. We observe that the states $|\Psi\rangle$ and $|\Psi'\rangle$ lead to two expectation values which differ by a gauge transformation of the fields within the expectation value. Note that the gauge transformation does not leave the Lorenz gauges

$$[\partial{\cdot}A(x), a^\dagger_{(\mathrm{G})}(\vec{p})] = i\partial^2 e^{-ip\cdot x} = 0. \tag{6.59}$$

Hence null states induce residual gauge transformation within the Lorenz gauges.

Now it appears that the choice of representative in an equivalence class has undesirable impact on certain expectation values. Gladly, this does not apply to gauge-invariant observables. For instance, the electromagnetic field strength is unaffected

$$[F_{\mu\nu}(x), a^\dagger_{(\mathrm{G})}(\vec{p})] = \partial_\mu(p_\nu e^{-ip\cdot x}) - \partial_\nu(p_\mu e^{-ip\cdot x}) = 0. \tag{6.60}$$

Moreover, the coupling of the gauge potential to a conserved current J^μ

$$J[A] = \int d^4x\, J^\mu(x) A_\mu(x) \tag{6.61}$$

commutes with $a^\dagger_{(\mathrm{G})}$

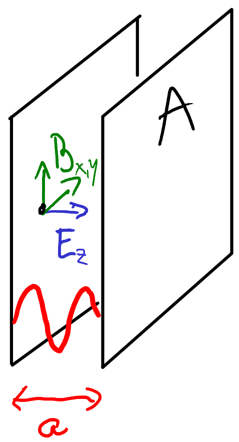
$$\begin{aligned}[][J[A], a^\dagger_{(\mathrm{G})}(\vec{p})] &= i\int d^4x\, J^\mu(x)\partial_\mu e^{-ip\cdot x}\\ &= -i\int d^4x\, e^{-ip\cdot x}\partial_\mu J^\mu(x) = 0.\end{aligned} \tag{6.62}$$

The expectation value of any gauge-invariant operator composed from $F_{\mu\nu}$, $J[A]$ or similar combinations thus does not depend on the choice of representatives, and it is consistent to define physical states as equivalence classes.

6.4 Casimir Energy

At this point we can already compute a quantum effect of the electromagnetic field, the Casimir effect. The Casimir effect is a tiny force between nearby conductors which exists even in the absence of charges, currents or medium. In classical electrodynamics no forces are expected. There are several alternative explanations for the quantum origin of the force. One is the exchange of virtual photons between the conductors. An equivalent explanation attributes the force to a change of vacuum energy of the electromagnetic field induced by the presence of the plates. The latter one has a quite efficient derivation, and we shall present it here.

Setup. We place two large planar metal plates at a small distance into the vacuum (much smaller than their size, but much larger than atomic distances). In our idealised setup, the plates extend infinitely along the x and y directions. They are separated by the distance a in the z direction. We will not be interested in the microscopic or quantum details of the metal objects. We simply assume that they are classical conductors and that they shield the electromagnetic field efficiently.



(6.63)

At the surface of the plates, the electric fields must be orthogonal $E_x = E_y = 0$ while the magnetic field must be parallel $B_z = 0$. In order to match these conditions simultaneously at both plates, the z-component of the wave vector (momentum) must be quantised

$$p_z \in \frac{\pi}{a}\mathbb{Z}. \tag{6.64}$$

Careful analysis shows that for $p_z = 0$ only one of the two polarisation vectors is permissible. Conversely, for $p_z \neq 0$ both polarisations are good. To achieve cancellations in this case, each wave must be synchronised to its reflected wave where $p_z \to -p_z$. Hence we should only count the contributions with $p_z > 0$.

Vacuum Energy. Just like the scalar field, the electromagnetic field carries some vacuum energy. The discretisation modifies the vacuum energy E_0, which results in a force between the plates if the new vacuum energy depends on the distance a.

The sum and integral of all permissible modes between the plates yields the energy

E per area A [10]

$$E = \int \frac{A\,dp_x\,dp_y}{(2\pi)^2}\left(\tfrac{1}{2}e(p_x,p_y,0) + 2\sum_{n=1}^{\infty}\tfrac{1}{2}e(p_x,p_y,\pi n/a)\right) \tag{6.65}$$

For convenience we shall exploit the rotation symmetry in the x-y-plane to simplify the expression to

$$\frac{E}{A} = \int_0^\infty \frac{p\,dp}{2\pi}\left(\tfrac{1}{2}p + \sum_{n=1}^{\infty}\sqrt{p^2+\pi^2 n^2/a^2}\right) \tag{6.66}$$

As discussed earlier, this expression diverges due to UV contributions at large momenta.

Regularisation. We also emphasised earlier that infinities are largely our own fault. The idealised setup was somewhat too ideal.

For macroscopic electromagnetic waves, we certainly made the right assumption of total reflection. But it is also clear that the conducting plates will behave differently for hard gamma radiation. This is precisely where the problems arise, so we seem to be on the right track. Electromagnetic waves with wave length much smaller than atomic distances or energies much larger than atomic energy levels will pass the conducting plates relatively unperturbed. These modes therefore should be discarded from the above sum.[11]

Therefore, we must introduce a UV cutoff for the modes. Define a function $f(e)$ which is constantly 1 for sufficiently small energies, constantly 0 for sufficiently large energy and which somehow interpolates between the 1 and 0 for intermediate energy. The cutoff replaces each contribution $\frac{1}{2}e$ by $\frac{1}{2}f(e)e$

$$\frac{E_{\mathrm{IR}}}{A} = \int_0^\infty \frac{p\,dp}{2\pi}\left(\tfrac{1}{2}pf(p) + \sum_{n=1}^{\infty}\sqrt{p^2+\pi^2 n^2/a^2}\,f(\sqrt{\ldots})\right). \tag{6.67}$$

Let us keep in mind the remaining contribution in the ultraviolet where we convert the sum to an integral due to the absence of quantisation in the z-direction

$$\frac{E_{\mathrm{UV}}}{A} = \int_0^\infty \frac{p\,dp}{2\pi}\int_0^\infty dn\sqrt{p^2+\pi^2 n^2/a^2}\,\bigl(1-f(\sqrt{\ldots})\bigr). \tag{6.68}$$

[10] A sum over the modes in some box of volume V in d dimensions turns into an integral over momenta when the volume is very large. In a box, the positive and negative modes are coupled, so the integral is over positive p only with integration measure $V\,d^dp/\pi^d$. In the absence of boundary contributions, the integration domain extends to positive and negative p which is compensated by the measure $V\,d^dp/(2\pi)^d$.

[11] The modes do contribute to the vacuum energy between the plates. Importantly, the distance between the plates will hardly enter their contribution, and consequently they cannot contribute to forces.

Summation. This expression is now finite, but certainly depends on the cutoff in $f(e)$. Let us write it as a sum of integrals

$$\frac{E_{\mathrm{IR}}}{A} = \tfrac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n), \tag{6.69}$$

with

$$F(n) = \int_0^\infty \frac{p\,dp}{2\pi}\sqrt{p^2+\pi^2n^2/a^2}\, f\big(\sqrt{p^2+\pi^2n^2/a^2}\big) \tag{6.70}$$

It is convenient to use energy as integration variable

$$e = \sqrt{p^2+\pi^2n^2/a^2}, \qquad p\,dp = e\,de, \tag{6.71}$$

and write the integral as

$$F(n) = \frac{1}{2\pi}\int_{\pi n/a}^\infty de\, e^2 f(e). \tag{6.72}$$

The Euler–MacLaurin summation formula writes the above sum for E_{IR}/A as an integral plus correction terms

$$\frac{E_{\mathrm{IR}}}{A} = \int_0^\infty dn\, F(n) - \sum_{k=1}^{\infty}(-1)^k \frac{\mathrm{B}_{2k}}{(2k)!}F^{(2k-1)}(0), \tag{6.73}$$

where we have used that the function $F(n)$ is constantly zero at ∞ due to the cutoff. Here B_n is the n-th Bernoulli number.

Let us analyse the two terms: The first term we can rewrite as

$$\frac{E_{\mathrm{int}}}{A} = \frac{1}{2\pi}\int_0^\infty dn \int_{\pi n/a}^\infty de\, e^2 f(e) = \frac{2a}{4\pi^2}\int_0^\infty de' \int_{e'}^\infty de\, e^2 f(e). \tag{6.74}$$

It depends on the cutoff, but it is manifestly linear in a. In fact, it combines nicely with the contribution from UV modes above the cutoff that we dropped earlier

$$\frac{E_0}{A} = \frac{E_{\mathrm{UV}}+E_{\mathrm{int}}}{A} = \frac{2a}{4\pi^2}\int_0^\infty de' \int_{e'}^\infty de\, e^2. \tag{6.75}$$

As such it represents the vacuum energy of the enclosed volume in the absence of plates. The same vacuum energy density is present outside the plates.[12] This term therefore does not contribute to the force because any shift of the plate would merely transfer some vacuum energy from the inside to the outside leaving the overall energy invariant. The fact that E_0 is formally infinite does not play a role. We therefore consider only the change in energy $\Delta E = E - E_0$ arising from the second term of the Euler–MacLaurin summation.

The second term can be evaluated near $n = 0$

$$F(n) = F(0) - \frac{1}{2\pi}\int_0^{\pi n/a} de\, e^2 f(e) = F(0) - \frac{\pi^2 n^3}{6a^3}\,, \tag{6.76}$$

[12]The factor of $2/2\pi$ is interpreted as follows: $1/2\pi$ is the correct measure for integration over p_z. Moreover, in the factor of 2 compensates for the restricted integration region $p_z \geq 0$.

where we used that $f(e) = 1$. Quite surprisingly, $F(n)$ is a polynomial with two terms. All cutoff dependence is in $F(0)$ which does not appear in the summation formula.[13] The single correction term contributes the following vacuum energy ($\mathrm{B}_4 = -1/30$)

$$\frac{\mathit{\Delta} E}{A} = -\frac{\mathrm{B}_4}{4!} F^{(3)}(0) = -\frac{\pi^2}{720 a^3}\,. \tag{6.77}$$

The presence of the conducting plates decreases the vacuum energy by some amount proportional to $1/a^3$.

Casimir Force. The Casimir force can be expressed as the pressure

$$P = \frac{F}{A} = \frac{\mathit{\Delta} E'(a)}{A} = \frac{\pi^2}{240 a^4}\,. \tag{6.78}$$

Some properties:

- Bringing the plates closer decreases the energy, hence the Casimir force is attractive.
- It increases with the fourth power of the inverse distance as the plates come closer.
- It is a quantum effect, and there are hidden factor of $\hbar$ and c. Due to the fourth-power behaviour it can nevertheless be detected at reasonable separations. It becomes relevant at micrometer distance.
- It does not depend on the coupling strength of the electromagnetic field or on the elementary charge.

6.5 Massive Vector Field

So far we have discussed the massless vector field. Among the UIR's of the Poincaré group there is also the massive representation with spin 1. Massive vector particles exist in nature as the $\mathrm{W}_\pm$ and Z_0 bosons transmitting the weak nuclear interactions.[14]

Lagrangian. We can add a mass term to the vector Lagrangian to obtain the corresponding quantum field

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu V_\nu \partial^\mu V^\nu + \tfrac{1}{2}\partial_\mu V_\nu \partial^\nu V^\mu - \tfrac{1}{2} m^2 V^\mu V_\mu. \tag{6.79}$$

The corresponding equation of motion reads

$$\partial^2 V_\mu - \partial_\mu \partial^\nu V_\nu - m^2 V_\mu = 0. \tag{6.80}$$

[13] The reason is apparently that the cutoff is in a region of energies where the difference between a sum and an integral does not matter.

[14] The implementation of interacting massive vector fields actually needs much more care. Interacting vectors fields can acquire mass only through the Higgs mechanism.

By taking the total derivative of this equation, we see that it implies the simpler equation $-m^2\partial^\mu V_\mu = 0$. Substituting this result in the original equation of motion then yields a system of two equations

$$\partial^2 V_\mu - m^2 V_\mu = 0, \qquad \partial^\mu V_\mu = 0. \tag{6.81}$$

The first equation is the Klein–Gordon equation for each component of V_μ, the second equation removes one of the four potential orientations. The degrees of freedom agree with the classification of UIR's.

Correlators. We now want to quantise this system. In the canonical approach we first derive the conjugate momenta

$$\varPi_\mu = \dot{V}_\mu - \partial_\mu V_0. \tag{6.82}$$

As before, there is no conjugate momentum for the field V_0 hinting at the presence of constraints. Constrained systems are somewhat tedious to handle in the Hamiltonian framework and therefore in canonical quantisation. Instead, let us take a shortcut. We consider the fields to be operators and cook up unequal-time commutation relations

$$[V_\mu(x), V_\nu(y)] = \varDelta^{\mathrm{V}}_{\mu\nu}(x-y). \tag{6.83}$$

Our previous experience has shown that correlators can be composed from derivatives acting on the correlator of the scalar field. This automatically implements the Klein–Gordon equation. Here we propose[15] [16]

$$\varDelta^{\mathrm{V}}_{\mu\nu}(x) = \bigl(\eta_{\mu\nu} - m^{-2}\partial_\mu\partial_\nu\bigr)\,\varDelta(x). \tag{6.84}$$

The combination of derivatives was constructed such that $\varDelta^{\mathrm{V}}$ satisfies the polarisation equations

$$\partial^\mu \varDelta^{\mathrm{V}}_{\mu\nu}(x) = \partial^\nu \varDelta^{\mathrm{V}}_{\mu\nu}(x) = 0. \tag{6.85}$$

Equal-Time Commutators. Next let us see what this proposal implies for the equal-time commutators. The non-vanishing ones read as follows

$$\begin{aligned}
[V_0(\vec{x}), V_k(\vec{y})] &= im^{-2}\partial_k\delta^3(\vec{x}-\vec{y}),\\
[V_0(\vec{x}), \dot{V}_0(\vec{y})] &= -im^{-2}\partial_k\partial_k\delta^3(\vec{x}-\vec{y}),\\
[V_k(\vec{x}), \dot{V}_l(\vec{y})] &= i\delta_{kl}\delta^3(\vec{x}-\vec{y}) - im^{-2}\partial_k\partial_l\delta^3(\vec{x}-\vec{y}),\\
[\dot{V}_0(\vec{x}), \dot{V}_k(\vec{y})] &= i\partial_k\delta^3(\vec{x}-\vec{y}) - im^{-2}\partial_k\partial_l\partial_l\delta^3(\vec{x}-\vec{y}).
\end{aligned} \tag{6.86}$$

These relations appear somewhat unusual since they mix time and space components of V_μ.

[15] The correlator $\varDelta_+$ and the propagator G take an equivalent form in terms of their scalar field counterparts.

[16] The factor of $1/m^2$ in not as innocent as it may appear. When adding interactions, this term involving an inverse mass scale actually makes the theory behave badly for large momenta.

Let us replace the time derivatives $\dot{V}_k$ by the associated conjugate momenta $\mathit{\Pi}_k$. For the spatial components we recover the canonical commutator

$$[V_k(\vec{x}), \mathit{\Pi}_l(\vec{y})] = i\delta_{kl}\delta^3(\vec{x}-\vec{y}). \tag{6.87}$$

The commutators involving V_0 and $\dot{V}_0$ can be recovered using the equations of motion. The latter actually give an explicit solution for the field V_0 and its time derivative $\dot{V}_0$

$$V_0 = -m^{-2}\partial_k \mathit{\Pi}_k, \qquad \dot{V}_0 = \partial_k V_k. \tag{6.88}$$

In other words, V_0 is not an elementary field and its commutation relations follow from the canonical one above

$$\begin{aligned}
[\dot{V}_0(\vec{x}), \mathit{\Pi}_k(\vec{y})] &= i\partial_k\delta^3(\vec{x}-\vec{y}),\\
[V_k(\vec{x}), V_0(\vec{y})] &= im^{-2}\partial_k\delta^3(\vec{x}-\vec{y}),\\
[V_0(\vec{x}), \dot{V}_0(\vec{y})] &= im^{-2}\partial_k\partial_k\delta^3(\vec{x}-\vec{y}).
\end{aligned} \tag{6.89}$$

Hamiltonian Framework. We have obtained a reasonable QFT framework for our massive scalar field. Now we can revisit the Hamiltonian framework. First we perform a Legendre transformation of the Lagrangian for spatial components of the fields V_k [17]

$$\begin{aligned}
H &= \int d^3\vec{x}\,\Big(\mathit{\Pi}_k\dot{V}_k - \mathcal{L}\Big)\\
&= \int d^3\vec{x}\,\Big(\tfrac{1}{2}\mathit{\Pi}_k\mathit{\Pi}_k + \tfrac{1}{2}m^{-2}\partial_k\mathit{\Pi}_k\partial_l\mathit{\Pi}_l\\
&\qquad\qquad + \tfrac{1}{2}\partial_k V_l\partial_k V_l - \tfrac{1}{2}\partial_l V_k\partial_k V_l + \tfrac{1}{2}m^2 V_k V_k\Big).
\end{aligned} \tag{6.90}$$

Here, we have also substituted the solution for the field V_0 and its time derivative.

We note that the Hamiltonian is slightly unusual in that it contains derivatives of the momenta along with inverse powers of the mass. The inverse powers of the mass in fact prevent us from taking the massless limit.[18]

Gladly, this Hamiltonian implies the desired equations of motion

$$\begin{aligned}
\dot{V}_k &= -\{H, V_k\} = \mathit{\Pi}_k - m^{-2}\partial_k\partial_l\mathit{\Pi}_l,\\
\dot{\mathit{\Pi}}_k &= -\{H, \mathit{\Pi}_k\} = \partial_l\partial_l V_k - \partial_k\partial_l V_l - m^2 V_k.
\end{aligned} \tag{6.91}$$

It is not at all obvious that these equations imply the Klein–Gordon equation. However, their twisted form is required to be able to solve for the field V_0 easily and thereby obtain the correct energy.

[17] The Hamiltonian is manifestly positive since $\tfrac{1}{2}\partial_k V_l\partial_k V_l - \tfrac{1}{2}\partial_l V_k\partial_k V_l = \tfrac{1}{4}(\partial_k V_l - \partial_l V_k)^2$.

[18] We may impose a gauge by demanding $\partial_k\mathit{\Pi}_k = -m^{-2}V_0 = 0$. This eliminates the inverse mass from the Hamiltonian and validates the massless limit. Using $\partial_k\mathit{\Pi}_k = -m^{-2}V_0$ the gauge also implies $\dot{V}_0 = \partial_k V_k = 0$, i.e. the gauge is the Coulomb gauge.