Sheet VIII

Due: week of December 3

Both questions deal with the first post-Newtonian approximation.

Question 1 [*Lie derivative of the metric*]: For the definition of V, x, X and e see the part of the lecture dealing with the first post-Newtonian approximation. Let

$$v^i = V^a \frac{\partial x^i}{\partial X^a}. \tag{1}$$

(i) From

$$e_{ij} = \frac{\partial X^a}{\partial x^i} \frac{\partial X^b}{\partial x^j} \delta_{ab} \tag{2}$$

deduce that

$$\frac{\partial e_{ij}}{\partial t} = v^k \frac{\partial e_{ij}}{\partial x^k} + e_{kj} \frac{\partial v^k}{\partial x^i} + e_{ik} \frac{\partial v^k}{\partial x^j}. \tag{3}$$

(ii) Show that (3) coincides with the Lie derivative of e given by

$$\left(\mathcal{L}_v e\right)_{ij}(x) := \frac{d}{ds} \left(f_s^* e\right)_{ij}(x)\Big|_{s=0}, \tag{4}$$

where $f_s^* e$ is the pullback of the metric e given by

$$\left(f_s^* e\right)_{ij}(x) = \frac{\partial f_s^m}{\partial x^i}(x) \frac{\partial f_s^n}{\partial x^j}(x) e_{mn}(f_s(x)) \tag{5}$$

and f_s is the flow of the vectorfield v, i.e. for $x_0 \in \mathbb{R}^3$ we define $f_s^m(x_0) := x^m(s)$ where $x(s)$ satisfies

$$\frac{dx^i}{ds} = v^i(x), \qquad x^i|_{s=0} = x_0^i. \tag{6}$$

Question 2 [*Cross term in the metric*]: As in the lecture we denote by ρ the energy density and by p the momentum density. We assume that

$$\frac{\partial \rho}{\partial t} = 0, \qquad \operatorname{supp} \rho = \Omega \tag{7}$$

for some $\Omega \subset\subset \mathbb{R}^3$. Show that to leading order in $1/r$, where $r := \sqrt{\sum_{i=1}^3 (x^i)^2}$, the cross term in the metric of the first post-Newtonian approximation can be expressed by

$$-\frac{4GL}{c^2 r} \sin^2 \theta dt d\phi, \tag{8}$$

where L is the total angular momentum defined by

$$L^k := \int_\Omega \varepsilon^k{}_{ij} x^i p^j d^3x \tag{9}$$

and r, θ and ϕ denote the usual spherical coordinates. Furthermore, the x^3-axis is chosen in the direction of L.

Hints:

(i) Use the equation of continuity and integrate

$$\frac{\partial}{\partial x^j}(x^i p^j) \tag{10}$$

over Ω', where $\Omega' \supset \overline{\Omega}$, to show that the total momentum vanishes, i.e.

$$P^i := \int_\Omega p^i d^3x = 0. \tag{11}$$

(ii) Now use

$$\alpha_i(x) = \frac{G}{2}\int_\Omega \left[\frac{7\delta_{ij}}{|x-x'|} + \frac{(x^i - x'^i)(x^j - x'^j)}{|x-x'|^3}\right] p^j(x') d^3x' \tag{12}$$

together with (11) to show that α vanishes to first order in $1/r$.

(iii) Define $N^i := x^i/r$ and show that to leading order in $1/r$

$$\frac{1}{|x-x'|} = \frac{1}{r}\left(1 + \frac{N \cdot x'}{r}\right). \tag{13}$$

(iv) Define

$$D^{ij} := \int_\Omega x^i p^j d^3x \tag{14}$$

and integrate

$$\frac{\partial}{\partial x^j}(x^i x^k p^j) \tag{15}$$

over Ω' to show that

$$D^{ki} + D^{ik} = 0 \tag{16}$$

and furthermore

$$D^{ij} = \frac{1}{2}\varepsilon^{ij}{}_k L^k. \tag{17}$$

(v) Use (13), (16) and (17) in (12) to show that to leading order in $1/r$

$$\alpha_i = \frac{2G}{r^2}\varepsilon_{ijk} L^j N^k. \tag{18}$$

(vi) From the lecture we know that the cross term in the metric is given by

$$-\frac{2}{c^2}\alpha_i dt dx^i. \tag{19}$$

Take the x^3-axis in the direction of L and introduce standard spherical coordinates r, θ and ϕ to show the result.