Sheet VII

Due: week of November 26

Question 1 [*Energy-momentum-stress tensor for electromagnetic field*]:

(i) Derive the energy-momentum-stress tensor for the electromagnetic field

$$T_E^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}(g^{-1})^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \tag{1}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor.

Hint: Use the definition

$$\frac{\partial}{\partial g_{\mu\nu}}(L_E\, d\mu_g) = -\frac{1}{2}T_E^{\mu\nu}\, d\mu_g, \tag{2}$$

where L_E is the electromagnetic Lagrangian

$$L_E = \frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta} \tag{3}$$

and $d\mu_g$ is the volume element

$$d\mu_g = \sqrt{-\det g}\, d^4x. \tag{4}$$

(ii) Let now

$$T^{\mu\nu} = T_M^{\mu\nu} + T_E^{\mu\nu}, \tag{5}$$

where T_M is the energy-momentum-stress tensor of matter. Use $\nabla_\nu T^{\mu\nu} = 0$ and the definition of the spacetime force f^μ

$$f^\mu := \nabla_\nu T_M^{\mu\nu} \tag{6}$$

to show that

$$f^\mu = F^{\mu}{}_{\nu}J^\nu, \tag{7}$$

where J^μ is the spacetime current. Now restrict to Minkowski spacetime. Find the time component f^0 and space components f^i in terms of E^i, B^i, ρ and J^i, where E^i is the electric field, B^i is the magnetic field, J^i is the electric current and ρ is the charge density.

Hint: Use Maxwell's equations

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu. \tag{8}$$

Question 2 [*Geodesics in 3+1 formulation*]: The geodesic equation can be derived as critical points of the action

$$L = \int_a^b \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda. \tag{9}$$

Suppose we use a maximal time function t as the parameter for the geodesic, i.e. $\lambda = t$. Show that in the limit as $c \to \infty$, up to addition of a constant and multiplication with a constant, (9) becomes the action of a particle of unit mass in the gravitational field with potential ψ.

Hint: The metric is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -c^2\phi^2 dt^2 + \overline{g}_{ij}dx^i dx^j, \tag{10}$$

where

$$\phi = 1 + c^{-2}\psi + \dots, \qquad \overline{g}_{ij} = \delta_{ij} + \dots \tag{11}$$

and $\dots$ denotes higher order terms.

Question 3 [*Kasner metric*]: We look at the Kasner metric

$$g = -dt^2 + \overline{g}, \tag{12}$$

where

$$\overline{g} = \sum_{i=1}^{3} t^{2p_i}(dx^i)^2 \tag{13}$$

and t is the temporal distance function from a given spacelike hypersurface

$$(g^{-1})^{\mu\nu}\partial_\mu t \partial_\nu t = -1. \tag{14}$$

(i) Calculate the components of the second fundamental form k_{ij}.

(ii) Calculate the components of the Ricci tensor $\overline{R}_{ij}$.

(iii) Show that (12) is a vacuum solution of the Einstein field equations if and only if

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1. \tag{15}$$

Hint: Use the Einstein field equations in 3+1 form once we chose a foliation of the level sets of the temporal distance function.

Question 4 [*Scalar curvature of maximal hypersurface*]: Suppose we have a maximal foliation. Look at the scalar curvature of a maximal hypersurface. Show that $\overline{R} \geq 0$ with $\overline{R} = 0$ at a point if and only if $k_{ij} = 0$ and $\rho = 0$ at this point.