Sheet V

Due: week of October 29

Question 1 [*Proper energy density of a plane wave*]: We look at a plane electromagnetic wave propagating in the positive x-direction in Minkowski spacetime. Let the electric field E and the magnetic field B lie in the y-z-plane and let the components of these fields be functions of $x - t$.

(i) Use the Maxwell equations to show that

$$B_y = -E_z, \qquad B_z = E_y. \tag{1}$$

(ii) Recall the definition of the proper energy density

$$\rho := \inf_{u \in H_1^+} T(u, u). \tag{2}$$

Show that ρ vanishes but that the infimum in (2) is not attained. Explain this result physically.

Hint: Take $u = (u^t, u^x, 0, 0)$ and look at the limit $u^x \to \infty$.

Question 2 [*Energy momentum stress tensor for electromagnetic field*]:

(i) We look at the eigenvalues of the tensor for the electromagnetic field $F_{\mu\nu}$, i.e. $\lambda \in \mathbb{C}$ satisfying

$$F_{\mu\nu} v^\nu = \lambda \, g_{\mu\nu} v^\nu, \tag{3}$$

where $v \in \mathbb{C}^4$ is the corresponding eigenvector. Show that if λ is an eigenvalue of F so is $-\lambda$ and $\overline{\lambda}$. Show that for the set of eigenvalues of $F_{\mu\nu}$ the following generic cases are possible:

(a) Four real eigenvalues: $0 < \lambda_1 < \lambda_2$, $\lambda_3 = -\lambda_1$, $\lambda_4 = -\lambda_2$.

(b) Four imaginary eigenvalues: $\mathrm{i}\mu_1$, $\mathrm{i}\mu_2$, $\mathrm{i}\mu_3 = -\mathrm{i}\mu_1$, $\mathrm{i}\mu_4 = -\mathrm{i}\mu_2$, where $\mu_1, \mu_2 \in \mathbb{R}$, $0 < \mu_1 < \mu_2$.

(c) Four complex eigenvalues: $\lambda + \mathrm{i}\mu$, $\lambda - \mathrm{i}\mu$, $-\lambda - \mathrm{i}\mu$, $-\lambda + \mathrm{i}\mu$, where $\lambda, \mu \in \mathbb{R}$.

(d) Two real and two imaginary eigenvalues: λ, $-\lambda$, $\mathrm{i}\mu$, $-\mathrm{i}\mu$, where $\lambda, \mu \in \mathbb{R}$.

Hint: Look at the determinant of the matrix given by the tensor

$$A_{\mu\nu}(\lambda) := F_{\mu\nu} - \lambda \, g_{\mu\nu} \tag{4}$$

and its transpose and recall that $F_{\mu\nu} = -F_{\nu\mu}$.

(ii) Show that only the case (d) is possible in the physical case of three space and one time dimension.

Hints:

(a) To real eigenvalues correspond real eigenvectors. Let L_1, L_2 be real eigenvectors corresponding to the eigenvalues λ_1, λ_2. Use

$$F_{\mu\nu}L_i^\mu L_i^\nu = 0, \qquad F_{\mu\nu}L_i^\mu L_j^\nu = -F_{\nu\mu}L_j^\nu L_i^\mu, \qquad i,j \in \{1,2\}, i \neq j \tag{5}$$

together with (3) to show that

$$g(L_i, L_i) = 0, \qquad g(L_i, L_j) = 0. \tag{6}$$

From this deduce the impossibility of case (a).

(b) To complex eigenvalues correspond complex eigenvectors. Let $M_1 = R_1 + \mathrm{i}S_1$ and $M_2 = R_2 + \mathrm{i}S_2$ be complex eigenvectors corresponding to the imaginary eigenvalues $\mathrm{i}\mu_1$, $\mathrm{i}\mu_2$ (R_i, S_i real vectors). Use (5) with M_i in the role of L_i together with (3) to show that

$$g(R_i, S_i) = 0, \qquad g(R_i, R_i) = g(S_i, S_i) \tag{7}$$

as well as

$$g(R_i, R_j) = g(S_i, S_j) = 0, \qquad g(R_i, S_j) = 0. \tag{8}$$

Consider the planes spanned by $\{R_1, S_1\}$ and $\{R_2, S_2\}$ and deduce the impossibility of case (b).

(c) Use the fact that if M is a complex eigenvector corresponding to the complex eigenvalue ν it follows that $\overline{M}$ is a complex eigenvector corresponding to the complex eigenvalue $\overline{\nu}$. Use (5) with M and $\overline{M}$ in the role of L_i and L_j. Similarly to the above cases show the impossibility of case (c).

(iii) Let F be given by

$$F = E\,dx \wedge dt + B\,dy \wedge dz, \tag{9}$$

i.e. $F_{xt} = E$, $F_{yz} = B$. Show that this form of F corresponds to the physical case (d) and find λ and μ. Show that

$$L_+ := \frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \qquad L_- := \frac{\partial}{\partial t} - \frac{\partial}{\partial x}, \qquad M := \frac{\partial}{\partial y} + \mathrm{i}\frac{\partial}{\partial z}. \tag{10}$$

are the corresponding eigenvectors. Conversly, show that if the generic physical case, i.e. case (d), holds, then we can suitably choose the Lorentz frame such that F is of the form (9). Find the components of the energy momentum stress tensor $T_{\mu\nu}$ where

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{11}$$