ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Statistical Physics
Exercise 11

HS 12
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Exercise 11.1 The Ising Model in the High-Temperature Limit

Consider the Ising model with nearest neighbor interactions in the presence of a homogeneous magnetic field h,

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j - h \sum_i S_i \ , \quad J > 0 \ , \tag{1}$$

where the spins assume the values $S_i = \pm S$ along the magnetic field and the sum $\sum_{\langle i,j \rangle}$ runs over nearest neighboring sites on the lattice. The number of spins is very large, $N \gg 1$, such that surface effects may be neglected.

a) Determine the partition function in the high-temperature limit $\beta J \ll 1$.
Hint: Note that for $\beta J \ll 1$, one may neglect bond-bond correlations and the partition function simplifies to

$$Z \approx \sum_{\{S_l\}} e^{\beta h \sum_i S_i} \left(1 + \frac{\beta J}{2} \sum_j \sum_{m \in \Lambda_j} S_j S_m \right) \ , \tag{2}$$

where Λ_j represents the set of nearest neighbors of site j such that $|\Lambda_j| = z$ with the coordination number z.

b) Calculate the spin susceptibility at $h = 0$. In analogy to the lecture notes (Section 3.4.6), plot $1/\chi_0$ as a function of temperature and extrapolate the high-T limit to lower temperatures to find the intersection on the T-axis. What is the physical interpretation of the intersection temperature?

Exercise 11.2 The Lattice Gas model

The lattice gas model is obtained by dividing the volume V into microscopic cells which are assumed to be small such that they contain at most one gas molecule. We neglect the kinetic energy of a molecule and assume that only nearest neighbours interact. The total energy is given by

$$H = -\lambda \sum_{\langle i,j \rangle} n_i n_j \tag{3}$$

where the sum runs over nearest-neighbour pairs and $\lambda > 0$ is the nearest-neighbour attraction. There is at most one particle in each cell ($n_i = 0$ or 1). This model is a simplification of hard-core potentials, like the Lennard-Jones potential, characterized by an attractive interaction and a very short-range repulsive interaction that prevents particles from overlapping.

a) Show the equivalence of the grand canonical ensemble of the lattice gas model with the canonical ensemble of an Ising model in a magnetic field.

In the following we will use the mean-field solution of the Ising model discussed in Chapter 5.2 of the lecture notes and the equivalence stated in a) in order to discuss the liquid-gas transition in the lattice gas model.

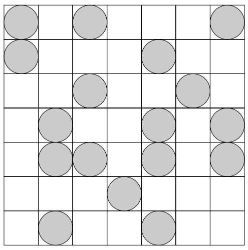

Figure 1: Schematic view of the lattice gas model.

b) Derive a self-consistence equation for the density $\rho = \langle n_i \rangle$ and discuss its solutions as a function of the temperature T and chemical potential μ.

c) Find the equation of state $p = p(T, \rho)$ or $p = p(T, v)$ and discuss the liquid-gas transition in the $p - v$ diagram. Thereby, $v = 1/\rho$ is the specific volume. Compare with the van der Waals equation of state:

$$\left(p + \frac{\tilde{a}}{v^2}\right)\left(v - \tilde{b}\right) = k_B T\,.$$

d) Find the phase diagram ($T - p$ diagram). Determine the phase boundary $(T, p_c(T))$ and, in particular, compute the critical point $(T_c, p_c(T_c))$.

Office Hours: Monday, December 3rd, 8:00-10:00 HIT K 23.7 (Evert van Nieuwenburg)