Statistical Physics
Exercise 10

Exercise 10.1 Bose–Einstein Condensation

a) In Section 4.5 of the lecture notes, we have derived an expression for the specific heat $C_V(T)$ of the spinless Bose gas both above and below the critical temperature T_c (Eq. 4.81). While C_V does not diverge at T_c, it has a cusp there (Fig. 4.4), suggesting that a T-derivative of C_V does diverge.

Evaluate

$$\Delta = \lim_{T \to T_c+} \partial_T C_V(T) - \lim_{T \to T_c-} \partial_T C_V(T) \neq 0$$

to show that $\partial_T C_V(T)$ is not continuous at T_c, and conclude that indeed $\partial_T^2 C_V(T)$ diverges at the critical temperature.

b) As we saw in a), in the vicinity of a phase transition several thermodynamic quantities may show non-analytic behavior. The way in which these quantities diverge gives us useful information about the transition itself. Oftentimes, one finds that some quantity X shows a power-law behavior near the critical temperature, i.e., $X(T) \propto (T - T_c)^\gamma$. The exponent γ is called the corresponding *critical exponent*.

Show that the compressibility κ_T of the spinless Bose gas satisfies a power law for $T \to T_c+$, and find the corresponding critical exponent.

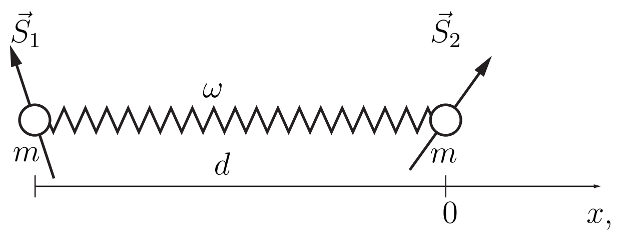
Hint: At $T = T_c$ we have $z = 1$. It is convenient to expand κ_T in the variable $\nu = \ln z$ about $\nu = 0$.

Exercise 10.2 Magnetostriction in a Spin-Dimer-Model

As in Exercise 8.2, we consider a dimer consisting of two spin-1/2 particles with Hamilton operator

$$\mathcal{H}_0 = J \left(\vec{S}_1 \cdot \vec{S}_2 + 3/4 \right)$$

and $J > 0$ (note that the energy levels are shifted as compared to Ex. 8.2). This time, however, the distance between the two spins is not fixed, but they are connected to a spring:



The spin–spin coupling constant depends on the distance between the two sites such that the Hamilton operator of the system is

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 + J(1 - \lambda\hat{x}) \left(\vec{S}_1 \cdot \vec{S}_2 + 3/4 \right) \qquad (1)$$

where $\lambda \geq 0$, m is the mass of the two constituents, $m\omega^2$ is the spring constant and where x denotes the displacement from the equilibrium distance d between the two spins (in the case of no spin-spin interaction).

a) Write the Hamiltonian (1) in second quantized form and calculate the partition sum, the internal energy, the specific heat and the entropy. Discuss the behavior of the entropy in the limit $T \to 0$ for different values of λ.

Hint: Set $\hbar = 1$ and introduce an observable $\hat{n}_t$ satisfying

$$\langle \sigma | \hat{n}_t | \sigma \rangle = \begin{cases} 1 & \text{if } \sigma \text{ is a triplet} \\ 0 & \text{if } \sigma \text{ is a singlet} \end{cases}$$

for any vector $|\sigma\rangle$ in the Hilbert space describing the spin part of the dimer.

b) Calculate the expectation value of the distance of the two spins, $\langle d + \hat{x} \rangle$, as well as the fluctuation, $\langle (d + \hat{x})^2 \rangle$. How are these quantities affected by a magnetic field in z-direction, i.e., by adding an additional term in (1) of the form

$$\mathcal{H}_m = -g\mu_B H \sum_i S_i^z \quad ?$$

c) If the two sites are oppositely charged, i.e., $\pm q$, the dimer forms a dipole with moment $P = q\langle d + x \rangle$. This dipole moment can be measured by applying an electric field E in x-direction,

$$\mathcal{H}_{\text{el}} = -q(d + \hat{x})E.$$

Calculate the zero-field susceptibility of the dimer,

$$\chi_0^{(\text{el})} = -\left.\frac{\partial^2 F}{\partial E^2}\right|_{E=0},$$

and compare your result with the fluctuation-dissipation theorem which asserts that

$$\chi_0^{(\text{el})} \propto \left\langle (d + \hat{x})^2 \right\rangle - \left\langle d + \hat{x} \right\rangle^2.$$

Plot the zero-field susceptibility as a function of the applied magnetic field H and discuss your result.

Office Hours: Monday, November 26th, 8–10 AM (Michael Walter, HIT K 31.5)