# Statistical Physics
# Exercise Sheet 9



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**Exercise 9.1 Ideal quantum gas in a harmonic trap**

In this exercise we compare the bosonic and fermionic quantum corrections to a spinless ideal gas confined in a three-dimensional harmonic potential with the classical case. For the results of the classical case see section 2.4.3 of the lecture notes.

The energy states of the gas are given by

$$E_{\mathbf{a}} = \hbar\omega(a_x + a_y + a_z) \ , \tag{1}$$

where we neglect the zero point energy of $E_0 = 3\,\hbar\omega/2$ as usual. The occupation number of the oscillator modes of the state $E_{\mathbf{a}}$ is given by $\mathbf{a} = (a_x, a_y, a_z)$ with $a_i \in \{0, 1, 2, ...\}$.

a) Consider the high-temperature, low-density limit ($z \ll 1$). Derive the grand canonical partition function $\mathcal{Z}_{b,f}$ of this system and compute the grand potential $\Omega_{b,f}$ for bosons and fermions. Show that

$$\Omega_f \propto f_4(z) \ , \qquad \Omega_b \propto g_4(z) \ , \tag{2}$$

where the functions $f_s(z)$ and $g_s(z)$ are defined as

$$f_s(z) = -\sum_{l=1}^{\infty} (-1)^l \frac{z^l}{l^s} \ , \qquad g_s(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^s} \ . \tag{3}$$

b) Derive the internal energy $U$ and the average particle number $\langle N \rangle$ of both the bosonic and the fermionic systems.

In order to get $U$ in terms of $N$ (instead of dealing with the chemical potential), introduce the parameter

$$\rho \equiv \left( \frac{\hbar\omega N^{1/3}}{k_B T} \right)^3 , \tag{4}$$

and relate it to $z$ using the high-temperature, low-density expansion of $\langle N \rangle$. Interpret the condition $\rho \ll 1$.

Then, expand $U$ up to second order in $\rho$, relating it to $N$.

c) Compute the specific heat $C$. Which quantity has to be fixed in order to do that?

Compute the thermal expansion coefficient $\alpha$ using effective volume $V_{\text{eff}}$ defined in terms of the average square displacement of the harmonic oscillator $\langle r^2 \rangle$ as $V_{\text{eff}} = 4\pi/3\, \langle r^2 \rangle^{3/2}$. Give an interpretation of $V_{\text{eff}}$.

d) Interpret your results for $U$, $C$, and $\alpha$ by comparing them with the corresponding results for the classical Boltzmann gas. How do the quantum corrections influence the bosonic and fermionic systems?

e) Find the critical temperature $T_c$ at which Bose-Einstein condensation occurs. How can this be reconciled with the high-temperature, low-density limit?

**Office hour:** Monday, November 19, 8-10 am (Sarah Etter, HIT K 12.2)