# Statistical Physics
# Exercise 6

HS 12
Prof. M. Sigrist

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## Exercise 6.1 The Ising Paramagnet

Consider $N$ localized non-interacting magnetic moments which take the values $s_i = \pm s$. In the presence of an external magnetic field $h$ the Hamiltonian of this system is given by

$$\mathcal{H} = -\sum_i h s_i. \tag{1}$$

Use the canonical ensemble to calculate the free energy $F(T,h)$, the caloric and thermal equations of state, the heat capacity $C(T,h)$ and the magnetic susceptibility $\chi(T,h)$. Find a relation between the fluctuations in the magnetization and the susceptibility.

## Exercise 6.2 Non-interacting Particles in the Gravitational Field

Consider a gas of non-interacting particles in the gravitational field

$$V_{\text{grav}}(x, y, z \geq 0) = mgz \tag{2}$$

with the gravitational constant $g > 0$ at fixed temperature $T$. The volume of the gas is confined to a vertical, cylindrical vessel (radius $R$) of semi-infinite height.

a) Using the canonical ensemble, find the Helmholtz free energy, the entropy, and the internal energy of this system.

b) Consider the system from the viewpoint of a local thermal equilibrium. Find the local particle density at height $z$, $n(z)$, normalized such that $N = \int \mathrm{d}^3q\, n(q)$. Find the local pressure $p(z)$ as well as the local internal energy density $u(z)$. Express $p(z)$ and $u(z)$ in terms of $n(z)$ to find the local caloric and thermal equations of state.

c) Calculate the heat capacity, interpret and compare the different results using:

   i) the entropy found in a).
   ii) the equipartition law for $U = \langle \mathcal{H} \rangle$.
   iii) the local caloric and thermal equations of state.
   iv) the variance $(\Delta\mathcal{H})^2$.

   *Hints for:*

   ii) Rewrite the Hamiltonian such that the equipartition law as given in chapter 3.4.2 in the lecture notes may be applied directly.
   iii) Keeping the specific volume $v(z) = N/n(z)$ constant is equivalent to a constant local density and thus the local version of (1.17) in the lecture notes combined with (1.44) is given by

   $$c_p = \left(\frac{\partial u}{\partial T}\right)_n + \left\{\left(\frac{\partial U}{\partial V}\right)_T + p\right\}\alpha = \left(\frac{\partial u}{\partial T}\right)_n + T\left(\frac{\partial p}{\partial T}\right)_n \alpha. \tag{3}$$

   First, show that even with the local viewpoint the thermal expansion coefficient $\alpha$ is given by $1/T$, independent of $z$.
   iv) Use 3.4.3 in the lecture notes and identify the kinetic and potential contributions to the variance. Define the total potential energy through $\langle z \rangle$.

**Office hour:** Monday, October 29, 8-10 am (Sarah Etter, HIT K 12.2)