Statistical Physics
Exercise Sheet 5

Exercise 5.1 Drude Conductivity

The goal of this exercise is to extend the calculation of the conductivity of an electron gas in the relaxation time approximation (as discussed in the lecture) to the case of a time-dependent electric field, $\vec{E}(t) = \vec{E}_0 \, e^{-i\omega t}$.

a) Split the distribution function into an equilibrium distribution $f_0(\vec{p})$ and a small perturbation $g(\vec{p}, t)$, i.e.

$$f(\vec{p}, t) = f_0(\vec{p}) + g(\vec{p}, t)\,, \tag{1}$$

and find the Boltzmann equation for $g(\vec{p}, t)$. Justify that the solution is of the form $g(\vec{p}, t) = g_\omega(\vec{p})\, e^{-i\omega t}$ and find $g_\omega(\vec{p})$. Furthermore, find expressions for the current response $\vec{j}_\omega$ and the conductivity σ_ω (defined by $\vec{j}_\omega = \sigma_\omega \vec{E}_0$).

b) Assume a Maxwell-Boltzmann distribution for $f_0(\vec{p})$ and calculate the time-dependent current $\vec{j}(t)$ for an external field given by $\vec{E}(t) = \vec{E}_0 \cos \omega t$. Show that in the limiting case $\omega \to 0$ the Drude conductivity is recovered (see lecture notes, sect. 2.7).

c) From the result of b), calculate

$$p = \langle \vec{j} \cdot \vec{E} \rangle_t = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \mathrm{d}t \; \vec{j}(t) \cdot \vec{E}(t)\,. \tag{2}$$

This term describes Ohmic heating (it is the local version of $P = UI$), which is accompanied by an increase in entropy. Observe that $H(t) = H(t + \frac{2\pi}{\omega})$. What are the assumptions made in a) and b)? How can you explain a non-vanishing p even though H is periodic?

d) The derivation of the Drude conductivity in a) and b) used some drastic approximations. Unfortunately, for most interesting systems, this is the usual state of affairs rather than an exception. Here we will derive a rather general criterion for checking a given approximation for a response function.
Begin by verifying that any complex function σ_ω which is analytic in the upper half complex plane and which fulfills $\sigma_{|\omega| \to \infty} = 0$ will satisfy the following relations

$$\mathrm{Re}\{\sigma_\omega\} = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \mathrm{d}\nu \, \frac{\mathrm{Im}\{\sigma_\nu\}}{\nu - \omega}\,, \tag{3}$$

$$\mathrm{Im}\{\sigma_\omega\} = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \mathrm{d}\nu \, \frac{\mathrm{Re}\{\sigma_\nu\}}{\nu - \omega} \tag{4}$$

with $\mathcal{P}$ denoting the Cauchy principal value. Verify that the expression which you found in a) for σ_ω together with the assumption that $f_0(\vec{p})$ be given by the Maxwell-Boltzmann distribution satisfies these expressions.
The relationships which you just proved are called the *Kramers-Kronig* relations

and are often used in practice both as a check for approximate response functions as they can be seen to be a consequence of causality, and also in situations in which only one component (real or imaginary) of a response function is available to recover the remaining one.

Exercise 5.2 Mircrocanonical Ensemble: Harmonic Potential with Uniform External Force

We consider an ideal gas of N non-interacting particles in a harmonic trap, described by the following Hamilton function:

$$\mathcal{H}(p,q) = \sum_{i=1}^{N} \left\{ \frac{\vec{p}_i^{\,2}}{2m} + \frac{D\vec{q}_i^{\,2}}{2} \right\} . \tag{5}$$

a) Using the microcanonical ensemble, calculate the entropy of this system and the caloric equation of state, i.e. $S(U,...)$ and $U(T,...)$, respectively.

   Hint: The volume of an n-dimensional sphere of radius R is given by $V(R) = C_n R^n$ with $C_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$, where $\Gamma(z+1) \approx (2\pi)^{1/2} e^{-z} z^{z+1/2}$ for $z \gg 1$ in the Stirling approximation.

b) Now assume that the particles have charge e (miraculously still not interacting with each other) and apply an external electric field $\vec{E}$ leading to the following correction to the Hamilton function:

$$\Delta\mathcal{H}(p,q) = -e \sum_{i=1}^{N} \vec{E} \cdot \vec{q}_i \, . \tag{6}$$

   Determine the entropy and the caloric equation of state of the gas including $\Delta\mathcal{H}(p,q)$. What is the dielectric polarization of system, $\vec{P}$, as a function of the electric field?

   Hint: Note that $dU = \cdots + \vec{P} d\vec{E}$.

Office Hours: Monday, October 22, 8-10 am (Juan Osorio, HIT G 32.4)