Dissipation in Quantum Systems Exercise 4

Exercise 4.1 Harmonic Oscillator I

Consider a harmonic oscillator consisting of a particle with mass m and coordinate x (the particle can move only along one direction). The eigenfrequency is given by ω_0. Since the harmonic oscillator is coupled to a thermal bath, a friction force affects the particle proportional to the particle's velocity. The corresponding relaxation time is τ.
In the lecture, it was shown that the response function $R_{xx}(t-t')$ is given by

$$R_{xx}(t-t') = \frac{1}{m\Omega} e^{-\gamma(t-t')} \sin\left(\Omega(t-t')\right) \cdot \theta(t-t') \tag{1}$$

with $\gamma = 1/2\tau < \omega_0$ and the normalized frequence $\Omega = \sqrt{\omega_0^2 - \gamma^2}$.

a) In the limit of strong damping ($\gamma > \omega_0$), show that the response function is given by

$$R_{xx}(t-t') = \frac{1}{m\tilde{\Omega}} e^{-\gamma(t-t')} \sinh\left(\tilde{\Omega}(t-t')\right) \cdot \theta(t-t') \tag{2}$$

where $\tilde{\Omega}$ is now defined as $\tilde{\Omega} = \sqrt{\gamma^2 - \omega_0^2}$.

b) Derive the correlation function $C_{xx}(t-t')$ for the strongly damped region by using the fluctuation-dissipation theorem,

$$k_B T \cdot R_{xx}(t) = -\frac{\partial C_{xx}(t)}{\partial t} \cdot \theta(t). \tag{3}$$

Hint: Due to the form of $R_{xx}(t)$, try the following ansatz

$$C_{xx}(t) = e^{-\gamma t}\left[A\cosh(\tilde{\Omega}t) + B\sinh(\tilde{\Omega}t)\right] \tag{4}$$

in order to solve the differential equation!

c) Show that the Fourier transform $R_{xx}(\omega)$ is given by

$$R_{xx}(\omega) = \frac{1}{2m\tilde{\Omega}}\left[\frac{1}{\gamma - \tilde{\Omega} - i\omega} - \frac{1}{\gamma + \tilde{\Omega} - i\omega}\right]. \tag{5}$$

d) Use the quantum fluctuation-dissipation theorem

$$C^q_{xx}(\omega) = \frac{2\hbar}{1 - \exp(-\beta\hbar\omega)} \cdot \mathrm{Im}\left[R_{xx}(\omega)\right] \tag{6}$$

and compute the quantum correlation function $C^q_{xx}(\omega)$! What is the quantum correlation function at $T = 0$?

e) Determine the value of $C^q_{xx}(t=0)$ at $T = 0$! Simplify the result in the limit $\gamma \gg \omega_0$ and show that

$$C^q_{xx}(t=0) = \frac{\hbar}{\pi m\gamma} \log\left(\frac{2\gamma}{\omega_0}\right). \tag{7}$$