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Dissipation in Quantum Systems Exercise 2

HS 12
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In Exercise 1 we saw that the Langevin equation is a powerful tool for the investigation a particle moving in a fluid by using a stochastic function $\xi(t)$, a microscopic quantity. However, there is another access to the problem based on macroscopic terminology, the Fokker-Planck equation.
Given a probability distribution $P(x_0, t_0)$ for the location of a single particle at $t = t_0$, the Fokker-Planck equation determines the evolution of $P(x, t)$ for $t > t_0$. A simple version of the Fokker-Planck equation is given by

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left[A(x,t)P(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[B(x,t)P(x,t)\right] \tag{1}$$

where $A(x,t)$ is denoted as the drift term and $B(x,t)$ is the diffusion term. In the following, we want to derive it for a simple model.

Exercise 2.1 Random Walk

We want to derive the Fokker-Planck equation and its solution for a simple model, the so-called random-walk model. This model consists of a particle moving in a (for simplicity) *one* dimensional lattice ($x_{i+1} - x_i = a$) and in discrete time steps t_j ($t_{j+1} - t_j = \tau$). At each time step the particle can hop with equal probability $p_\rightarrow = p_\leftarrow = p = 1/2$ either to the left hand or to the right hand side, see Fig. 1.

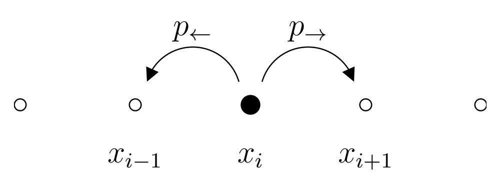

Figure 1: Hopping of a particle from x_i to x_{i-1} with probability $p_\leftarrow$ or to x_{i+1} with $p_\rightarrow$.

The probability distribution $P(x_i, t_j; x_0, t_0)$ of the particle satisfies the conditions

$$P(x_i, t_0; x_0, t_0) = \delta_{x_i, x_0} \tag{2}$$

$$\sum_{x_i} P(x_i, t_j; x_0, t_0) = 1 \tag{3}$$

where Eq. (2) fixes the initial condition at $t_j = t_0$ and Eq. (3) is due to particle number conservation.
It is not hard to see that the time evolution is given by

$$\begin{aligned} P(x_i, t_j + \tau; x_0, t_0) &= P(x_{i-1}, t_j; x_0, t_0) \cdot p_\rightarrow + P(x_{i+1}, t_j; x_0, t_0) \cdot p_\leftarrow \\ &= \frac{1}{2}\big(P(x_{i-1}, t_j; x_0, t_0) + P(x_{i+1}, t_j; x_0, t_0)\big). \end{aligned} \tag{4}$$

a) Verify that P fulfills the equation

$$\left[\partial_t^\tau - \frac{a^2}{2\tau}\Delta_a\right] P(x_i, t_j; x_0, t_0) = 0 \tag{5}$$

where the operators ∂_t^τ and Δ_a are defined as

$$\partial_t^\tau f(x_i, t_j; \dots) = \frac{f(x_i, t_{j+1}; \dots) - f(x_i, t_j; \dots)}{\tau} \tag{6}$$

$$\Delta_a f(x_i, t_j; \dots) = \frac{1}{a^2}\big[f(x_{i+1}, t_j; \dots) + f(x_{i-1}, t_j; \dots) - 2f(x_i, t_j; \dots)\big]. \tag{7}$$

b) Show that $P(x_i, t_j; x_0, t_0)$ is given by

$$P(x_i, t_j; x_0, t_0) = \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi}\big(\cos ka\big)^{t_j - t_0/\tau} e^{ik(x_i - x_0)} \tag{8}$$

by solving Eq. (4).

Hint: Work in Fourier space and use

$$P(x_i, t_j; x_0, t_0) = \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi} P(k, t_j; x_0, t_0) e^{ikx_i} \tag{9}$$

where $P(k, t_0; x_0, t_0)$ is defined by

$$P(k, t_j; x_0, t_0) = \sum_{x_i} P(x_i, t_j; x_0, t_0) e^{-ikx_i}. \tag{10}$$

c) Calculate the continuum limit $(a \to 0, \tau \to 0)$ of Eq. (5) and (8) provided that $a^2/2\tau \equiv D$ is kept constant.

Hint: Expand $\cos x \approx 1 - x^2/2$ and use the identity $e^x = \lim_{N\to\infty}(1 + x/N)^N$.

d) Now let's assume that there is an inbalance in the hopping, i.e. we have $\lambda > 0$ such that

$$p_\rightarrow = \frac{1}{2}(1 + \lambda), \qquad p_\leftarrow = \frac{1}{2}(1 - \lambda). \tag{11}$$

Introduce the parameter $c = \gamma a/\tau$ and find the corresponding partial differential equation (in the continuum limit) for this case!

Solve the differential equation using the ansatz

$$P_{new}(x, t; x_0, t_0) = P(x - f(t), t; x_0, t_0) \tag{12}$$

where P is the solution of (c).