ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Dissipation in Quantum Systems
Exercise 1

HS 12
Dr. R. Chitra

Exercise 1.1 The Langevin Model

The simple Langevin equation

$$\ddot{x} + \frac{\dot{x}}{\tau} = \xi(t) \tag{1}$$

describes a free particle at position $x(t)$ which is subject to a random external force "$m\xi(t)$". Thus, it describes Brownian motion, the apparently random movement of a particle in a fluid due to collisions with the molecules of the fluid.
Equation (1) is a so-called *stochastic* differential equation because the external force $\xi(t)$ is purely random. Therefore, it should not appear in any physical quantity at the end of the day and one should average $\langle \ldots \rangle_\xi$ the corresponding physical quantity over an ensemble of ξ's.
The definition of the average is rather technical (see Exercise 1.2) but it is sufficient to know that it fulfills the conditions

$$\langle \xi(t) \rangle_\xi = 0, \qquad \langle \xi(t_1)\xi(t_2) \rangle_\xi = \nu_\xi \cdot \delta(t_1 - t_2). \tag{2}$$

Recapitulate the lecture notes and calculate the correlation functions

$$\langle \Delta x^2(t) \rangle_\xi \equiv \left\langle \left[x(t) - x(0)\right]^2 \right\rangle_\xi, \quad \text{and} \quad \langle v(t)v(0) \rangle_\xi \tag{3}$$

and verify the general relation

$$\lim_{t\to\infty} \left(\frac{\langle \Delta x^2(t) \rangle_\xi}{2t} \right) = \int_0^\infty dt \; \langle v(t)v(0) \rangle_\xi. \tag{4}$$

Hint: In order to calculate the Green's function, solve the equation

$$\ddot{G} + \frac{\dot{G}}{\tau} + \omega_0^2 G = \delta(t) \tag{5}$$

and take the limit $\omega_0 \to 0$ at the end.

Exercise 1.2 *** The Probability Distribution ***

The average $\langle A(\xi(t)) \rangle_\xi$ of a function A is defined as the functional integral

$$\langle A(\xi) \rangle_\xi \equiv N \int \mathcal{D}\left[\xi(t)\right] \; P[\xi] \cdot A(\xi(t)), \qquad P[\xi] \propto \exp\left[-\frac{1}{2\nu_\xi} \int_{-\infty}^{\infty} dt \; \xi^2(t)\right] \tag{6}$$

where N is a normalization factor and $P[\xi]$ is the probability distribution. Show that

$$\langle \xi(t) \rangle_\xi = 0, \qquad \langle \xi(t_1)\xi(t_2) \rangle_\xi = \nu_\xi \cdot \delta(t_1 - t_2). \tag{7}$$