1. World-sheet supersymmetry (intermediate)

An action with global worldsheet supersymmetry is given by

$$S = -\frac{1}{4\pi\kappa^2}\int d^2\xi(\partial^\alpha X^\mu\,\partial_\alpha X_\mu + \bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu).$$

The Grassmann-valued fields ψ^μ are two-dimensional Majorana spinors. The ρ^α are 2×2 γ-matrices satisfying

$$\{\rho^\alpha,\rho^\beta\} = 2\eta^{\alpha\beta}$$

and the Dirac conjugate is $\bar{\psi} = i\psi^\dagger\rho^0$ with representation

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

a) Show that this action is invariant under $\mathcal{N}=1$ supersymmetry

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \quad \delta\psi^\mu = \rho^\alpha\partial_\alpha X^\mu\epsilon.$$

Why are the components of ψ and ϵ Weyl "spinors"?

b) Evaluate the commutators $[\delta_1,\delta_2]X^\mu$ and $[\delta_1,\delta_2]\psi^\mu$ to show that the commutator of two supersymmetry transformations amount to a translation along the world-sheet.

c) Derive the Noether current (supercurrent) of supersymmetry transformations. Meditate on the relation between the supercurrent and the energy-momentum tensor.

2. Grande Finale: The super-Particle (intermediate)

A massless supersymmetric particle in D-dimensional Minkowski spacetime is described by the action

$$S_\ell = \int d\tau\left(\frac{\dot{X}^\mu\dot{X}_\mu}{2e} + \frac{i\dot{X}^\mu\psi_\mu\chi}{e} - i\psi^\mu\dot{\psi}_\mu\right)$$

where e is the einbein and χ its fermionic partner. ψ^μ are 2D Majorana fermions.

a) Derive the equations of motion for $X^\mu(\tau)$, $\psi^\mu(\tau)$ as well as e and χ. Find the correct supersymmetry transformations for e and χ by requiring that the action is invariant under (local!) supersymmetry transformations with

$$\delta X^\mu = i\epsilon\psi^\mu \qquad \text{and} \qquad \delta\psi^\mu = \frac{1}{2e}(\dot{X}^\mu - i\chi\psi^\mu)\epsilon.$$

b) Argue that you can set $e=1$ and $\chi=0$. Write down the gauged action. Important: What are the constraint equations that follow from this procedure? Interpret them!

c) Consider the (global) supersymmetry transformations (derive them from the ones given in a)) and check that the commutator of two supersymmetry transformations will result in a τ translation by an amount $\delta\tau$.