8 String Scattering

Compute a string scattering amplitude. Two methods:

- worldsheet junction(s). string cylinders with cuts. integration over junctions.
- vertex operators. integration over punctures locations.

8.1 Vertex Operators

State-operator map:
- Which operator creates a string?
- How to specify the momentum $q$?
- How to specify the string modes?

Solution is related to the operator $O[q] = \exp(iq\cdot X)$: Why?
- Momentum eigenstate: phase for translation $\exp(iq\cdot \epsilon)$.

Compute OPE with stress-energy $T$

$$T(z)O[q](w, \bar{w}) = \frac{1}{4}\kappa^2 q^2 O[q](w, \bar{w}) + \frac{\partial O[q](w, \bar{w})}{z-w} + \ldots$$

Primary operator with weights $(\frac{1}{4}\kappa^2 q^2, \frac{1}{4}\kappa^2 q^2)$!
- non-trivial, non-integer weight,
- quantum effect $\sim \kappa^2$.

Consider two-point correlator

$$\langle O_1[q_1]O_2[q_2]\rangle \simeq |z_1 - z_2|^{\kappa^2(q_1 \cdot q_2)}.$$ 

In fact, zero mode $X^\mu = x^\mu + \ldots$ contributes extra factor

$$\int d^Dx \exp(iq_1 \cdot x + iq_2 \cdot x) \sim \delta^D(q_1 + q_2).$$

Hence compatible with primary of weight $(\frac{1}{4}\kappa^2 q^2, \frac{1}{4}\kappa^2 q^2)$

$$\langle O_1[q_1]O_2[q_2]\rangle \simeq \frac{\delta^D(q_1 + q_2)}{|z_1 - z_2|^{\kappa^2 q^2}}.$$
Operator $\mathcal{O}[q](z, \bar{z})$ creates a string state at $(z, \bar{z})$. Worldsheet location unphysical, integrate:

$$V[q] = g_s \int d^2 z \mathcal{O}[q](z, \bar{z}).$$

Can only integrate weight $(1, 1)$ primary operators. Hence:

- mass $M^2 = -q^2 = -4/\kappa^2$; string tachyon!
- intercept $a = \bar{a} = 1$ due to worldsheet integration.

What about excited strings? Level-1 corresponds to

$$V^{\mu\nu}[q] = g_s \int d^2 z \partial X^\mu \bar{\partial} X^\nu \mathcal{O}[q].$$

- weight is $(1 + \frac{1}{4}\kappa^2 q^2, 1 + \frac{1}{4}\kappa^2 q^2) = (1, 1)$ for massless $q$.
- primary condition removes unphysical polarisations.
- gauge d.o.f. are total derivatives.

Vertex operator picture:

- CFT vacuum is empty worldsheet (genus 0, no punctures).
- $\int d^2 z \mathcal{O}[q](z, \bar{z})$ is string vacuum $|0; q\rangle$ (add puncture).
- $\int d^2 z \ldots \mathcal{O}[q](z, \bar{z})$ are excited string states. Insertions of $\partial^n X^\mu$ correspond to string oscillators $\alpha^\mu_n$, insertions of $\bar{\partial}^n X^\mu$ correspond to $\bar{\alpha}^\mu_n$.

8.2 Veneziano Amplitude

Consider $n$-point amplitude (with $\mathcal{O}_k = \mathcal{O}[q_k](z_k, \bar{z}_k)$)

$$A_n \sim \frac{1}{g_s^2} \langle V_1 \ldots V_n \rangle \sim g_s^{n-2} \int d^2 z \langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle$$

- simplest to use tachyon vertex operators,
- can do others, but add complications (fields),
- computation & result qualitatively the same.

Perform Wick contractions and zero mode integration

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle \sim \delta^D(Q) \prod_{j<k} |z_j - z_k|^{\kappa^2 q_j \cdot q_k}.$$

Integral invariant under Möbius transformations ($q_k^2 = 4/\kappa^2$). Map three punctures to fixed positions $z_1 = \infty$, $z_2 = 0$, $z_3 = 1$. Remaining integral for $n = 4$ strings

$$A_4 \sim g_s^2 \delta^D(Q) \int d^2 z \, |z|^{\kappa^2 q_2 \cdot q_4} |1 - z|^{\kappa^2 q_1 \cdot q_4}$$

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can be performed

\[ A_4 \sim g_s^2 \delta^{D}(Q) \frac{\Gamma(-1 - \kappa^2 s/4) \Gamma(-1 - \kappa^2 t/4) \Gamma(-1 - \kappa^2 u/4)}{\Gamma(+2 + \kappa^2 s/4) \Gamma(+2 + \kappa^2 t/4) \Gamma(+2 + \kappa^2 u/4)}. \]

Mandelstam invariants:

\[ s = (q_1 + q_2)^2, \quad t = (q_1 + q_4)^2, \quad u = (q_1 + q_3)^2, \]

with relation \( s + t + u = -q_1^2 - q_2^2 - q_3^2 - q_4^2 = -16/\kappa^2. \)

This is the Virasoro-Shapiro amplitude for closed strings. Corresponding amplitude for open strings

\[ A_4 \sim g_s \frac{\Gamma(-1 - \kappa^2 s) \Gamma(-1 - \kappa^2 t)}{\Gamma(+2 + \kappa^2 u)} \]

was proposed (not calculated) earlier by Veneziano. Considered birth of string theory (dual resonance model).

Amplitudes have many desirable features:

- Poles at \( s, t, u = (N - 1)/\kappa^2 \) or \( s, t = (N - 1)/\kappa^2, \) virtual particles with string mass exchanged.
- Residues indicate spin \( J = 2N \) or \( J = N. \) Regge trajectory!
- Soft behaviour at \( s \to \infty. \) Even for gravitons!
- Manifest crossing symmetry \( s \leftrightarrow t \leftrightarrow u \) or \( s \leftrightarrow t. \) Amazing!

Not possible for QFT with finitely many particles.

### 8.3 String Loops

Result exact as far as \( \alpha' \) is concerned. Free theory in \( \alpha'! \)

However, worldsheet topology matters. String loop corrections for adding handles: higher genus. Power of \( g_s \) reflects Euler characteristic of worldsheet.

**Tree Level.** Worldsheet is sphere or disk with \( n \) punctures. Euler characteristic \(-2 + n \) or \(-1 + n/2.\) 6 global conformal symmetries, integration over \( n - 3 \) points.
**One Loop.** Worldsheets are tori with $n$ punctures. Euler characteristic $n$. 2 moduli: integration over Teichmüller space. 2 shifts; integration over $n-1$ points. $2n$ integrations; result: elliptic & modular functions; feasible!

**Two Loops.** Worldsheets are 2-tori with $n$ punctures. Euler characteristic $2+n$. 6 moduli, no shifts: $2(n+3)$ integrations. Hard, but can be done. No higher-loop results available.