

**Exercise 3.1 Relaxation of Polarized Spins Coupled to a Bath**

Consider a system  $S$  of  $N^{(S)}$  units with two states (spins  $\uparrow^{(S)}$  and  $\downarrow^{(S)}$ ) that are coupled to a reservoir  $R$  (infinitely larger than  $S$ ) consisting of other two-state units (labeled as  $\uparrow^{(R)}$  and  $\downarrow^{(R)}$ ). This is a simplified model for nuclear spins in a metal, whereby the nuclear spins represent our system and the electrons (electron spins) the reservoir.

We assume that the microstates in  $S$  and  $R$  are non-degenerate with an energy difference  $\Delta\epsilon$ ,

$$E_{\uparrow}^{(S)} - E_{\downarrow}^{(S)} = E_{\uparrow}^{(R)} - E_{\downarrow}^{(R)} = \Delta\epsilon, \quad (1)$$

where  $E_{\uparrow}^{(S)}$  and  $E_{\downarrow}^{(S)}$  ( $E_{\uparrow}^{(R)}$  and  $E_{\downarrow}^{(R)}$ ) are the energies for the two states in  $S$  ( $R$ ).

The reservoir  $R$  is in its thermodynamic equilibrium such that the occupation probabilities  $w_{\uparrow}^{(R)}$  and  $w_{\downarrow}^{(R)}$  of the two states satisfy the relation,

$$\frac{w_{\uparrow}^{(R)}}{w_{\downarrow}^{(R)}} = \exp\left\{-\frac{\Delta\epsilon}{k_B T^{(R)}}\right\}, \quad (2)$$

where  $T^{(R)}$  defines the temperature of  $R$ . At the initial time,  $t = 0$ ,  $S$  shall not be in equilibrium with  $R$ .

There is a coupling between the units of  $S$  and  $R$  in the following way. A spin of  $S$  can only be reversed, if simultaneously a spin of  $R$  is flipped in the opposite way, i.e:  $\uparrow^{(S)} \rightarrow \downarrow^{(S)}$  together with  $\downarrow^{(R)} \rightarrow \uparrow^{(R)}$  and  $\downarrow^{(S)} \rightarrow \uparrow^{(S)}$  together with  $\uparrow^{(R)} \rightarrow \downarrow^{(R)}$ . This corresponds to spin conservation as well as energy conservation.

Both types of processes occur with the same probability  $\Gamma$  (time reversal invariance):

$$|\dots, \uparrow, \dots\rangle^{(S)} \otimes |\dots, \downarrow, \dots\rangle^{(R)} \xrightarrow{\Gamma} |\dots, \downarrow, \dots\rangle^{(S)} \otimes |\dots, \uparrow, \dots\rangle^{(R)} \quad (3)$$

$$|\dots, \downarrow, \dots\rangle^{(S)} \otimes |\dots, \uparrow, \dots\rangle^{(R)} \xrightarrow{\Gamma} |\dots, \uparrow, \dots\rangle^{(S)} \otimes |\dots, \downarrow, \dots\rangle^{(R)} \quad (4)$$

From this consideration the effective transition rates ( $\Gamma_{\uparrow\downarrow}$ ,  $\Gamma_{\downarrow\uparrow}$ ) for the spins in  $S$  depend on the probabilities of finding states  $\uparrow^{(R)}$  and  $\downarrow^{(R)}$  in  $R$ , such that

$$\Gamma_{\uparrow\downarrow}^{(S)} = \Gamma w_{\downarrow}^{(R)}, \quad \Gamma_{\downarrow\uparrow}^{(S)} = \Gamma w_{\uparrow}^{(R)}. \quad (5)$$

- a) Formulate the master equation for the system  $S$  where  $N_{\uparrow}^{(S)}$  and  $N_{\downarrow}^{(S)}$  denote the occupation numbers of the two spin states ( $N_{\uparrow}^{(S)} + N_{\downarrow}^{(S)} = N^{(S)}$ ). What does the condition of detailed balance imply?
- b) Compute the time dependent polarization of the localized spin system, defined by  $P^{(S)}(t) = (N_{\uparrow}^{(S)}(t) - N_{\downarrow}^{(S)}(t))/N^{(S)}$ , for the following starting conditions at  $t = 0$ :
  - (i) unpolarized case ( $N_{\uparrow}^{(S)} = N_{\downarrow}^{(S)}$ ),
  - (ii) almost complete polarized in  $\downarrow$ -direction ( $N_{\downarrow}^{(S)} = N^{(S)}(1 - \delta)$ ,  $N_{\uparrow}^{(S)} = N^{(S)}\delta$ ,  $\delta \ll 1$ ),

- (iii) almost complete polarized in  $\uparrow$ -direction ( $N_{\uparrow}^{(S)} = N^{(S)}(1 - \delta)$ ,  $N_{\downarrow}^{(S)} = N^{(S)}\delta$ ,  $\delta \ll 1$ )

Determine the relaxation time for these three cases. Find the temperature of the spin system  $T^{(S)}$  for the three initial conditions (the temperature for  $S$  is defined similar to the temperature of the reservoir, Eq. (2)). How do you explain the sign of  $T^{(S)}$ ?

- c) Calculate the  $H$ -function of  $S$  and its derivative with respect to time for the three cases in b). Moreover, calculate the energy exchange between  $S$  and  $R$  as a function of  $t$  (take the temperature of the reservoir  $T^{(R)}$  as a parameter) for the same three cases. How does the temperature  $T^{(S)}$  evolve as a function of time? Calculate the specific heat of  $S$  as a function of temperature  $T^{(S)}$ .

### Exercise 3.2 The Linear Boltzmann Equation

We want to study the Boltzmann equation when only one-particle scattering is considered, contrary to two-particle scattering as presented in the lecture. Let us consider a system of particles imbedded in a static background of impurities such that one particle with the momentum  $\vec{p}$  has a certain probability to be scattered to a new state with the momentum  $\vec{p}'$ . We assume that the scattering processes are instantaneous, elastic and isotropic.

- a) Following the approach of the lecture notes, derive the Boltzmann equation of the system.
- b) List the quantities which are conserved in the one-particle collision and verify the relation

$$\int d^3p \varphi(\vec{p}) \left( \frac{\partial f(\vec{r}, \vec{p}, t)}{\partial t} \right)_{\text{coll}} = 0, \quad (6)$$

where  $\varphi(\vec{p})$  denotes a conserved quantity (we use the notations of the lecture notes).

- c) Prove the "H-theorem" discussing the assumptions you make. Find the equilibrium state and give the physical interpretation of a relaxation process towards equilibrium (what does it change in the system?).

**Office Hours:** Monday, October 10, 8 – 10 am (Andrei Plamada, HIT E 41.2)