### Handy properties of von Neumann entropy

1. Definition: $H(A)_\rho = -\mathrm{Tr}\big(\rho \log \rho\big) = -\sum_i \lambda_i \log \lambda_i$, where:
   (a) $\{\lambda_i\}_i$ are the eigenvalues of $\rho$;
   (b) the logarithm is $\log_2$;
   (c) to evaluate the entropies, $0 \log 0 = 0$;
   (d) nonation: we sometimes see just $H(A)$ or even $H(\rho)$.
2. Positivity: $H(A)_\rho \geq 0$ (because $0 \leq \lambda_i \leq 1$).
3. Entropy of pure states: $H(A)_{|\psi\rangle} = 0$ (because the density matrix has a single eigenvalue 1 for eigenvector $|\psi\rangle$).
4. Basis independence: $H(A)_\rho = H(A)_{U\rho U^\dagger}$ for unitaries $U$, because the eigenvalues are not affected by a change of basis.
5. Conditional entropy: $H(A|B)_\rho = H(AB)_\rho - H(B)_\rho$.
6. Strong subadditivity: $H(A|BC)_\rho \leq H(A|B)_\rho$. In other words, knowing more cannot hurt.

### Exercise 9.1 Some properties of von Neumann entropy

In this exercise you have to prove some more properties of von Neumann entropy. The first one is rather surprising: if two systems share a pure state, then the entropy of each of the systems is the same, independently of their dimensions. In other words, if you have a pure state $|\psi\rangle$ in a system represented by the hilbert space $\mathcal{H}$, then you can decompose the system in two parts, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, in any way you want and the entropy of $A$ will always be equal to the entropy of $B$, even if you choose to split $\mathcal{H}$ in a way such that $|\mathcal{H}_A| \ll |\mathcal{H}_B|$.

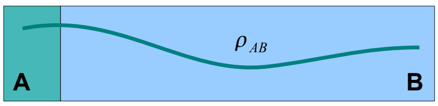


Figure 1: If $\rho_{AB}$ is pure $H(A)_\rho = H(B)_\rho$, independently of dimensions of subsystems $A$ and $B$.

To prove this, try writting a Schmidt decomposition of $|\psi\rangle$ (page 27 of the script).
The next property studies two systems that are in a product state, $\rho_{AB} = \rho_A \otimes \rho_B$. The systems are independent of each other—whatever operations or measurements you perform on $A$ will not affect $\rho_B$ and vice-versa. In this non-correlated case one would expect that the uncertainty about the global state is just the sum of the uncertainty about the two local subsystems—and, for once, quantum mechanics respects common sense, with $H(AB) = H(A) + H(B)$.
To prove that property, you may start by expanding the reduced states in their eigenbases,

$$\rho_A = \sum_k \gamma_k |k\rangle\langle k|_A, \qquad \rho_B = \sum_\ell \lambda_\ell |\ell\rangle\langle \ell|_B.$$

Now expand the composed state $\rho_{AB} = \rho_A \otimes \rho_B$ in those bases and compute its entropy directly.
In part *b*) we look at a special category of bipartite states, those that are classical on one of the subsystems. These states are introduced on pages 34–35 of the script. They have the form

$$\rho_{ZA} = \sum_z p_z |z\rangle\langle z|_Z \otimes \rho_A^z \tag{1}$$

for a fixed basis $\{z_Z\}_z$ of the first subsystem $\mathcal{H}_Z$ and a probability distribution $\{p_z\}_z$.
It help to look at one example of such a state. Consider two qubits, the computational basis and the classicaly correlated state

$$\rho_{ZA} = p\ |0\rangle\langle 0| \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + (1-p)\ |1\rangle\langle 1| \otimes \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

Actually, the first system can be a classical bit, since no cross terms like $|0\rangle\langle 1|$ appear there. The reduced state of system $A$ is just

$$\rho_A = \mathrm{Tr}_Z(\rho_{ZA}) = p\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + (1-p)\ \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix},$$

and in general, for a hybrid classical-quantum state of the form of Eq. 1,

$$\rho_A = \sum_z p_z\ \rho_A^z.$$

The reduced state of the classical system is

$$\rho_Z = \mathrm{Tr}_A(\rho_{ZA}) = p(\alpha+\delta)\ |0\rangle\langle 0| + (1-p)(\alpha'+\delta')\ |1\rangle\langle 1|,$$

or, in general,

$$\rho_z = \sum_z p_z \mathrm{Tr}(\rho_A^z)\ |z\rangle\langle z|.$$

These hybrid states may be interpreted as "state $\rho_A^z$ was prepared on system $A$ with probability $p_z$, and in that case the classical register $Z$ shows the value $z$, i.e., it is in the pure state $|z\rangle$." A measurement on system $Z$ performed in basis $\{z\}_z$ would allow us to determine which $\rho_A^z$ had been prepared, because the total state would became $|z\rangle\langle z| \otimes \rho_A^z$. Since in that case the reduced state of $A$ would be $\rho_A^z$, we call that "the state of system $A$ conditioned on the measurement outcome $z$ of system $Z$", $\rho_A^z = \rho_{A|Z=z}$.
Let us now go back to the exercise. You are asked to prove that for states like that of Eq. 1,

$$\begin{aligned} H(AZ) &= H(Z)_\rho + \sum_z p_z\ H(A|Z=z) \\ &= H(Z)_\rho + \sum_z p_z\ H(A)_{\rho_A^z}. \end{aligned}$$

I suggest that you expand the matrices $\rho_A^z$ in their eigenbases, for instance

$$\rho_A^z = \sum_k \lambda_k^z\ |k_z\rangle\langle k_z|.$$

If you now write $\rho_{ZA}$ using those expressions for $\rho_A^z$ and compute its entropy, you should get the desired result.
I won't help you in part $b)$ 2. Part $b)$ 3. asks you to show that for these states $H(Z|A) \geq 0$. One trick that may help is to imagine a system $Y$ that is just a copy of $Z$ and a state

$$\rho_{ZAY} = \sum_k p_z |z\rangle\langle z| \otimes \rho_A^z \otimes |y\rangle\langle y|.$$

You may check that the entropy of this state is the same than that of $\rho_{AB}$. In fact, you can show that $H(ZAY) = H(ZA)$ and $H(Z) = H(Y)$. Now use strong subaddivity to show what you want.

**Exercise 9.2 Majorisation and entanglement catalysis**

**Typo on the exercise sheet:** in Exercise 9.2 *c*), we should have

$$|\psi\rangle_{AB} = \sqrt{\mathbf{0.4}}\ |00\rangle + \sqrt{\mathbf{0.4}}\ |11\rangle + \sqrt{0.1}\ |22\rangle + \sqrt{0.1}\ |33\rangle,$$

with $\sqrt{0.4}$ instead of $\sqrt{4}$.
To learn more about majorization, check Section 5.3 of the script (p. 55 of the latest version), and this book by Nielsen and Vidal: `http://www.rintonpress.com/journals/qic-1-1/vidal.pdf` .
Quick recap: say that $\rho$ and $\sigma$ are $d$-dimensional states with eigenvalues $\{a_i\}_i$ and $\{b_i\}_i$, respectively. Then $\text{spec}(\rho) \prec \text{spec}(\tau)$ means that

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i, \forall k \leq d.$$

In part *a*) you just have to apply this to the qubit case, $d = 2$, and see the consequences for the Bloch vectors of the two states. Express the eigenvalues of $\rho$ (and $\sigma$) as a function of $|\vec{r}|$ (and $|\vec{t}|$) and the result should be direct.
For part *b*) 3. we apply a (von Neumann) projective measurement on $\rho$. The post-measurement state, not conditioned on the outcome, is just $\rho' = \sum_k P_k \rho P_k$, with $\sum_k P_k^\dagger P_k = \mathbb{1}$. We consider only orthonormal projectors, so $P_k^\dagger = P_k^2 = P_k$. Note also that $P_k P_\ell = P_\ell P_k = \delta_{k\ell} P_k$. An example of such a POVM is just to measure in an orthonormal basis, or a coarse-graining of that measurement, like $P_1 = \sum_{x=1}^{5} |x\rangle\langle x|, P_2 = \sum_{x=6}^{8} |x\rangle\langle x|$, for an o.n. basis $\{|x\rangle\}_{x=1}^{8}$.
We want to show that $\rho' \prec \rho$. Here is a suggestion on how to prove it. Say there are $n$ projectors $\{P_k\}_k$ in total. Create a family of operators $U_1, U_2, \ldots, U_n$ defined as

$$U_j = \sum_{k=1}^{n} \text{Exp}\left[2\pi i\ \frac{jk}{n}\right] P_k,$$

and check that they are unitaries. Now see that

$$\sum_j \sum_k U_j \rho U_j^\dagger = n\rho'.$$

Finally, use Corollary 5.3.3 from the script to prove the desired result.
For an alternative proof, start by proving the statement for two orthogonal projectors, and then use induction to obtain the general case.
Part *c*) is pretty straight-forward, and pretty strange! **Extra:** Can you think of an explicit procedure that Alice and Bob may use to take $|\psi\rangle|\tau\rangle \to |\phi\rangle|\tau\rangle$ via LOCC?

**Exercise 9.3 Information measures bonanza**

Quantum smooth entropies and mutual information look pretty much like their classical counterparts in the non-conditional case. Wait for conditional smooth entropies. *evil laugh* ... They're not that bad. *evil laugh* ... Really.
I will leave you to solve this exercise without help. *evil laugh* *choke*