Quantum Information Theory
Series 10

Exercise 10.1 Properties of the Fidelity

The fidelity is defined (for normalized states) as:

$$F(\rho,\sigma) := \left|\left|\sqrt{\rho}\sqrt{\sigma}\right|\right|_1 .$$

The generalized fidelity (used for subnormalized states), is defined as:

$$\overline{F}(\rho,\sigma) := \left|\left|\sqrt{\rho \oplus (1-\mathrm{Tr}\rho)}\sqrt{\sigma \oplus (1-\mathrm{Tr}\sigma)}\right|\right|_1 .$$

a) Given that $\rho, \sigma \in S_=(\mathcal{H})$ (where $S_=(\mathcal{H}) := \{\rho \in \mathcal{L}(\mathcal{H}) | \rho \geq 0, \mathrm{Tr}(\rho) = 1\}$, the set of all normalized states in the hilbert space $\mathcal{H}$), prove that the fidelity cannot decrease under CPTP maps $\mathcal{E}$:

$$F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho,\sigma).$$

b) Given that $\rho, \sigma \in S_\leq(\mathcal{H})$ (where $S_\leq(\mathcal{H}) := \{\rho \in \mathcal{L}(\mathcal{H}) | \rho \geq 0, \mathrm{Tr}(\rho) \leq 1\}$, the set of all subnormalized states in the hilbert space $\mathcal{H}$), prove that the generalized fidelity cannot decrease under projections Π:

$$\overline{F}(\Pi\rho\Pi, \Pi\sigma\Pi) \geq \overline{F}(\rho,\sigma).$$

Exercise 10.2 The Data Processing Inequality *(6 points)*

Strong subadditivity is a very important property in quantum information theory. In can be stated (for the von Neumann entropy) as:

$$H(A|B)_\rho \geq H(A|BC)_\rho. \tag{1}$$

a) The data processing inequality is a fundamental aspect of information theory, and is closely related to strong subadditivity. It states that if you perform any quantum operation (data processing) on a system B you cannot gain any more information about another quantum system A. Argue why Eq. 1 actually implies the data processing inequality:

$$H(A|B)_\rho \leq H(A|B')_{\rho'}, \text{ where } \rho'_{AB'} = [\mathcal{I} \otimes \mathcal{E}]\,(\rho_{AB}), \tag{2}$$

and $\mathcal{E}$ is a CPTP map.

Now you will prove a property of the smooth min-entropy, and then specialize this inequality to the von Neumann entropy. The conditional min-entropy can be defined as:

$$H_{\min}(A|B)_\rho := \max_\lambda \{\lambda \in \mathbb{R} \mid \exists\, \sigma_B \in S_=(\mathcal{H}_B) \text{ s.t. } \rho_{AB} \leq 2^{-\lambda} \mathbb{1}_A \otimes \sigma_B\}.$$

This version of the definition can be useful, and in particular in this exercise. The ϵ-smooth conditional min-entropy is defined as:

$$H^\epsilon_{\min}(A|B)_\rho := \max_{\rho' \in \mathcal{B}^\epsilon(\rho)} H_{\min}(A|B)_{\rho'},$$

where $\mathcal{B}^\epsilon(\rho) := \{\rho' \in S_\leq(\mathcal{H}) | F(\rho,\rho') \geq 1-\epsilon\}$, *i.e.* ρ' is ϵ-close to ρ in the fidelity (also $\mathcal{B}^\epsilon(\rho)$ is called an ϵ-ball around ρ).

b) Prove the following property of the smooth min-entropy:

$$H^\epsilon_{\min}(A|B)_\rho \geq H^\epsilon_{\min}(A|BC)_\rho.$$

You can use Lemma 6.5.3 from the script (which is the same inequality with $\epsilon = 0$).

c) Prove the chain rule:

$$H^\epsilon_{\min}(AB)_\rho - H^\epsilon_{\max}(B)_\rho \leq H^{3\epsilon}_{\min}(A|B)_\rho,$$

where $H^\epsilon_{\max}(B)_\rho := \min_{\tilde{\rho}\in\mathcal{B}^\epsilon(\rho)} \log|\mathrm{supp}\tilde{\rho}_B|$. **Hint:** Consider the projector onto the support of the state that optimizes $H^\epsilon_{\max}(B)$ and apply this to the definition of $H^\epsilon_{\min}(AB)$. Then use Exercise 10.1 and Uhlmann's theorem.

d) Prove the *quantum* asymptotic equipartition property (QAEP) for the smooth min-entropy,

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} H_{\min}^{\epsilon}(A^n)_{\rho^{\otimes n}} = H(A)_{\rho}.$$

Hint: Remember the classical asymptotic equipartition property you proved in Exercise 3.1 (where the smoothing was done in the trace distance), and Exercise 6.1 part c) (which relates the trace distance and the fidelity).

e) The QAEP can also be shown for $H_{\max}^{\epsilon}$. Use this fact and part b) and c) to show that

$$H(A|B)_{\rho} \leq \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} H_{\min}^{\epsilon}(A^n|B^n)_{\rho^{\otimes n}}. \tag{3}$$

f) Inequality (3) can also be shown to be true in the other direction, and therefore these quantities are actually equal. Use this fact to prove the following inequality for the von Neumann entropy:

$$H(A|B)_{\rho} \geq H(A|BC)_{\rho}.$$