Quantum Information Theory
Series 8

HS 11
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Exercise 8.1 Measurements as unitary evolutions

Consider a measurement on a system $\mathcal{H}_A$, whose output is in $\mathcal{H}_B$ that is described by the observable $M = \sum_{x \in \mathcal{X}} x P_x$, where $\{P_x\}_x$ are projectors. Suppose we want to measure a state, ρ_A, with this measurement. We can represent the measurement as a unitary evolution with an output on a larger system, $\mathcal{H}_B \otimes \mathcal{H}_R$, followed by a partial trace over R.

a) Show that $\mathcal{E}(\rho_A)$ can be written as unitary followed by a partial trace over S. This task can be broken down into the following steps:

i) What is the Kraus operator representation of the measurement of the operator M?

ii) If we write the projectors as $P_x = \sum_{k \in S_x} |k\rangle\langle k|$ (where S_x is the set of indices k that we sum over, which is different for each x, and $\{|k\rangle\}_k$ is an orthonormal basis), what is the Choi-Jamiołkowski (CJ) state?

iii) Find a purification of the CJ state. Label the purifying system with R.

iv) Apply the inverse of the CJ isomorphism to the purified state in (iii), and show that it is of the form $U\rho_A U^*$, where U is an isometry. The inverse CJ isomorphism is the map that takes a state $\rho_{A'BR}$ as input, and outputs a map $\mathcal{F}$. Specifically:

$$\mathcal{F}(\rho_A) = |\mathcal{H}_A| \mathrm{Tr}_{A'} \left(\left(\sum_{i,j} |i\rangle_{A'} \langle j|_A \rho_A |i\rangle_A \langle j|_{A'} \right) \otimes \mathbb{1}_{BR} \cdot \rho_{A'BR} \right),$$

where $\{|i\rangle\}_i$ is an orthonormal basis for A and A' (similarly for $\{|j\rangle\}_j$), and $\rho_{A'BR}$ is the CJ state purified on the system R.

v) Show that $\mathrm{Tr}_R(\mathcal{F}(\rho_A))$ has the same output as the measurement description in (i).

vi) Finally, explain how you could represent the measurement as a unitary followed by a partial trace over R.

b) Give an explicit expression for the map $\mathcal{E}$ for two different measurement on a qubit state described by the POVMs:

1. $\mathcal{M}_1 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.
2. $\mathcal{M}_2 = \{p|0\rangle\langle 0|, p|1\rangle\langle 1|, (1-p)\mathbb{1}_2\}$. What is the physical interpretation of this POVM?

Exercise 8.2 Entanglement and Teleportation

Imagine that Alice (A) has a pure state $|\psi\rangle_S$ of a system S in her lab. She wants to send the state to Bob, who lives on the Moon, but she does not want to pay the expensive price to ship it to the moon. We will see that if Alice and Bob share an entangled state, Alice can "teleport" the state $|\psi\rangle$ to the system B that Bob controls.
Formally, we have three systems $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. In this exercise we will assume all three are qubits. The initial state is

$$|\psi\rangle_S \otimes \frac{1}{\sqrt{2}} \left(|0_A 0_B\rangle + |1_A 1_B\rangle \right), \tag{1}$$

i.e. S is decoupled from A and B and these two are fully entangled in a Bell state. We may write $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

a) In a first step, Alice will measure systems S and A jointly in the Bell basis,

$$\left\{ \begin{array}{ll} \frac{1}{\sqrt{2}} \left(|0_S 0_A\rangle + |1_S 1_A\rangle \right), & \frac{1}{\sqrt{2}} \left(|0_S 0_A\rangle - |1_S 1_A\rangle \right), \\ \frac{1}{\sqrt{2}} \left(|0_S 1_A\rangle + |1_S 0_A\rangle \right), & \frac{1}{\sqrt{2}} \left(|0_S 1_A\rangle - |1_S 0_A\rangle \right) \end{array} \right\}. \tag{2}$$

Alice then (classically) communicates the result of her measurement to Bob. What is the reduced state of Bob's system (B) for each of the possible outcomes?

b) Depending on the outcome of the measurement by Alice, Bob may have to perform certain unitary operations on his qubit so that he recovers $|\psi\rangle$. Which operations are these?

c) Suppose that Alice does not manage to tell Bob the outcome of her measurement. Show that in this case he does not have any information about his reduced state and therefore does not know which operation to apply in order to obtain $|\psi\rangle$.

d) Show that this method of quantum teleportation also works for mixed states ρ_S.

e) There is no reason why the state ρ_S cannot be entangled with some other system that Alice and Bob do not control. Consider a purification of ρ_S on a reference system R,

$$\rho_S = \mathrm{Tr}_R |\phi\rangle\langle\phi|_{SR}. \tag{3}$$

Show that if you apply the quantum teleportation protocol on $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, not touching the reference system, the final state on $\mathcal{H}_B \otimes \mathcal{H}_R$ is $|\phi\rangle$.

This implies that quantum teleportation preserves entanglement — it simply transfers it from S and R to B and R.