Quantum Information Theory
Series 9

Definitions: von Neumann entropy

In this series we will derive some useful properties of the von Neumann entropy: the quantum version of Shannon entropy. We will also look at the strangeness of quantum mutual information. Before we start, here are a few definitions.
The von Neumann entropy of a density operator $\rho \in \mathcal{S}(\mathcal{H}_A)$ is defined as

$$H(A)_\rho = -\mathrm{Tr}\big(\rho \log \rho\big) = -\sum_i \lambda_i \log \lambda_i, \tag{1}$$

where $\{\lambda_i\}_i$ are the eigenvalues of ρ.
Given a composite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ we write $H(AB)_\rho$ to denote the entropy of the reduced state of a subsystem, $\rho_{AB} = \mathrm{Tr}_C(\rho_{ABC})$. When the state ρ is obvious from the context we drop the indices.
The *conditional* von Neumann entropy is defined as

$$H(A|B)_\rho = H(AB)_\rho - H(B)_\rho. \tag{2}$$

In the Alice-and-Bob picture this quantifies the uncertainty that Bob (who holds the B part of the quantum state ρ_{AB}) still has about Alice's state.
The strong sub-additivity property of the von Neumann entropy is very useful. It applies to a tripartite composite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$,

$$H(A|BC)_\rho \leq H(A|B)_\rho. \tag{3}$$

Exercise 9.1 Some properties of von Neumann entropy

a) Prove the following general properties of the von Neumann entropy:

1. If ρ_{AB} is pure, then $H(A)_\rho = H(B)_\rho$.
2. If two systems are independent, $\rho_{AB} = \rho_A \otimes \rho_B$, then $H(AB)_\rho = H(A)_{\rho_A} + H(B)_{\rho_B}$.

b) Consider a bipartite state that is classical on subsystem Z: $\rho_{ZA} = \sum_z p_z |z\rangle\langle z|_Z \otimes \rho_A^z$ for some basis $\{|z\rangle_Z\}_z$ of $\mathcal{H}_Z$. Show that:

1. The conditional entropy of the quantum part, A, given the classical information Z is
$$H(A|Z)_\rho = \sum_z p_z\, H(A|Z=z), \tag{4}$$
where $H(A|Z=z) = H(A)_{\rho_A^z}$.
2. The entropy of A is concave,
$$H(A)_\rho \geq \sum_z p_z\, H(A|Z=z). \tag{5}$$
3. The entropy of a classical probability distribution $\{p_z\}_z$ cannot be negative, even if one has access to extra quantum information, A,
$$H(Z|A)_\rho \geq 0. \tag{6}$$
Remark: Eq (6) holds in general only for classical Z. Bell states are immediate counterexamples in the fully quantum case.

Exercise 9.2 Majorisation and entanglement catalysis

a) *Warm-up.* Let ρ and τ be two single-qubit states characterized by their Bloch vectors,

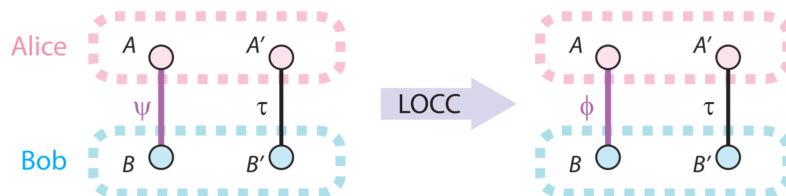
$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}), \qquad \tau = \frac{1}{2}(\mathbb{1} + \vec{t} \cdot \vec{\sigma}).$$

Show that $\operatorname{spec}(\rho) \prec \operatorname{spec}(\tau)$ if and only if $|r| \leq |t|$. Here, $\operatorname{spec}(\rho)$ denotes the spectrum of ρ, i.e., the set of eigenvalues of ρ.

b) We perform a projective measurement described by the POVM $\{P_i\}_i$ on a state ρ. Denote the post-measurement state (not conditioned on the outcome) by ρ'. Show that $\operatorname{spec}(\rho') \prec \operatorname{spec}(\rho)$.

c) We saw that two parties that share an initial bipartite pure state $|\psi\rangle_{AB}$ can transform it into another state $|\phi\rangle_{AB}$ via LOCC if and only if $\S(\psi) \prec \S(\phi)$ (Theorem 5.3.4). Here, $\S(\psi) = \operatorname{spec}(\operatorname{Tr}_A |\psi\rangle\langle\psi|)$ is the vector formed by the eigenvalues of the reduced density matrices on each side.

There are situations where, even if that condition is not satisfied, having access to an extra entangled state $|\tau\rangle_{A'B'}$ (a *catalyst*) allows the players to transform $|\psi\rangle$ into $|\phi\rangle$ and return the catalyst untouched in the end.



Consider the following three bipartite states (on a four-level system on each side),

$$|\psi\rangle_{AB} = \sqrt{0.4}\,|00\rangle + \sqrt{0.4}\,|11\rangle + \sqrt{0.1}\,|22\rangle + \sqrt{0.1}\,|33\rangle, \qquad |\phi\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.25}\,|11\rangle + \sqrt{0.25}\,|22\rangle,$$
$$|\tau\rangle_{A'B'} = \sqrt{0.6}\,|00\rangle + \sqrt{0.4}\,|11\rangle.$$

Check that $\S(\psi) \prec \S(\phi)$ does **not** hold, but $\S(\psi \otimes \tau) \prec \S(\phi \otimes \tau)$ does.

(Note that to compute $\S(\psi \otimes \tau)$ you have to trace out A and A' to obtain the reduced density matrix on Bob's side (BB').)

Exercise 9.3 Information measures bonanza

Take a system A in state ρ. Non-conditional quantum min- and max-entropies are given by

$$H_{\min}(A)_\rho = -\log \max_{\lambda \in \operatorname{spec}(\rho)} \lambda, \qquad H_{\max}(A)_\rho = \log \operatorname{rank}(\rho).$$

For instance, if ρ_A has eigenvalues $\operatorname{spec}(\rho_A) = \{0.6, 0.2, 0.2, 0\}$, we have $H_{\min}(A)_\rho = -\log 0.6$ and $H_{\max}(A)_\rho = \log 3$. The mutual information measures correlations between two systems. For ρ_{AB}, we have

$$\begin{aligned} I(A:B)_\rho &= H(A)_\rho + H(B)_\rho - H(AB)_\rho \\ &= H(A)_\rho - H(A|B)_\rho. \end{aligned}$$

a) Show that if $\operatorname{spec}(\rho) \prec \operatorname{spec}(\tau)$, then the entropy of ρ is larger than or equal to the entropy of τ, for the von Neumann, min- and max-entropies.

b) Consider two qubits A and B in a joint state ρ_{AB}.

1. Prove that the mutual information of the Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is maximal for a two-qubit system. This is why we say that Bell states are *maximally entangled*.
2. Show that $I(A:B) \leq 1$ for classically correlated states, $\rho_{AB} = p|0\rangle\langle 0|_A \otimes \sigma_B^0 + (1-p)|1\rangle\langle 1|_A \otimes \sigma_B^1$ (where $0 \leq p \leq 1$).

c) Show that if the bipartite state $|\psi\rangle_{AB}$ can be transformed into $|\phi\rangle$ via LOCC (without catalysts), then $I(A:B)_\psi \geq I(A:B)_\phi$.