Quantum Information Theory
Series 11

Exercise 11.1 Resource inequalities: teleportation and classical communication

We saw a protocol, teleportation, to transmit one qubit using two bits of classical computation and one ebit, $\substack{\overset{2}{\rightarrow} \\ \underset{1}{\Upsilon}} \geq \overset{1}{\rightsquigarrow}$ (Section 5.1, page 52 of the script). Now suppose that Alice and Bob share unlimited entanglement: they can use up as many ebits as they want. Can Alice send n qubits to Bob using less than $2n$ bits of classical communication? In other words, we want to know if the following is possible:

$$\substack{\overset{m}{\rightarrow} \\ \underset{\infty}{\approx}} \geq \substack{\overset{n}{\rightsquigarrow} \\ \underset{\infty}{\approx}}, \qquad m < 2n.$$

Prove that this is not the case. **Hint:** use superdense coding.

Exercise 11.2 A sufficient entanglement criterion

In general it is very hard to determine if a state is entangled or not. In this exercise we will construct a simple entanglement criterion that correctly identifies all entangled states in low dimensions.
Recall that we say that a bipartite state ρ_{AB} is separable (not entangled) if

$$\rho = \sum_k p_k \, \sigma_k \otimes \tau_k, \quad \forall k : p_k \geq 0, \sigma_k \in \mathcal{S}_=(\mathcal{H}_A), \tau_k \in \mathcal{S}_=(\mathcal{H}_B), \quad \sum_k p_k = 1.$$

a) Let $\Lambda_A : \mathrm{End}(\mathcal{H}_A) \mapsto \mathrm{End}(\mathcal{H}_A)$ be a positive map. Show that $\Lambda_A \otimes \mathcal{I}_B$ maps separable states to positive operators.
This means that if we apply $\Lambda_A \otimes \mathcal{I}_B$ to a bipartite state ρ_{AB} and obtain a non-positive operator, we know that ρ_{AB} is entangled. In other words, this is a sufficient criterion for entanglement.

b) Now we have to find a suitable map Λ_A. Show that the transpose,

$$\mathcal{T}\left(\sum_{ij} a_{ij} |i\rangle\langle j|\right) = \sum_{ij} a_{ji} \; |i\rangle\langle j|,$$

is a positive map from $\mathrm{End}(\mathcal{H}_A)$ to $\mathrm{End}(\mathcal{H}_A)$, but is not completely positive.

c) Apply the partial transpose, $\mathcal{T}_A \otimes \mathcal{I}_B$, to the ε-noisy Bell state

$$\rho^\varepsilon_{AB} = (1-\varepsilon) \; |\psi^-\rangle\langle\psi^-| + \varepsilon \; \frac{\mathbb{1}_4}{4}, \quad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad \varepsilon \in [0,1].$$

For what values of ε can we be sure that ρ^ε is entangled?
Remark: Indeed, it can be shown that the PPT criterion (positive partial transpose) is necessary and sufficient for systems of dimension 2×2 and 2×3.

Exercise 11.3 Properties of squashed entanglement

We defined squashed entanglement as

$$E_{sq}(A : B)_\rho = \frac{1}{2} \inf_E I(A : B|E)_\rho.$$

a) Show that the conditional mutual information can only decrease under local operations, i.e., $I(A : B|E)_\rho \geq I(A' : B'|E)_{\rho'}$, where $\rho'_{A'B'E} = [\mathcal{E}_{A \mapsto A'} \otimes \mathcal{F}_{B \mapsto B'} \otimes \mathcal{I}_E](\rho_{ABE})$, and $\mathcal{E}, \mathcal{F}$ are TPCPMs.
We call this the data processing inequality for the mutual information. What does it imply for squashed entanglement?

b) Prove that squashed entanglement is zero for separable states.