Eidgenössische Technische Hochschule Zürich
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# Quantum Information Theory
# Series 6

HS 11
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### Exercise 6.1 Distance bounds

In this exercise we'll bring together some of the concepts and techniques you have been learning over the past three weeks: purification, fidelity and trace distance.

The idea here is that we want to create a maximally entangled state $|\Psi\rangle$ between two systems $A$ and $A'$. We need this state to perform a cool quantum task (say for instance teleportation, which we will introduce later in the lecture). However, we are not sure we can create exactly $|\Psi\rangle$. We do know that we can create a state in $A \otimes B$ such that $B$ has almost no information about $A$: $\rho_{AB} \approx \mathbb{1}_A/|A| \otimes \rho_B$. How can this help us find an (approximately) maximally entangled state between $A$ and $A'$? And what is the probability that something might go wrong with out cool quantum task?

This may look like a contrived setting, but it's actually quite common that we can find a state like $\rho_{AB}$ (e.g. using decoupling; stick to Advanced Topics in the next semester to learn more about it!).

*a*) Given a trace-preserving quantum operation $\mathcal{E}$ and two states $\rho$ and $\sigma$, show that

$$\delta\left(\mathcal{E}(\sigma), \mathcal{E}(\rho)\right) \leq \delta(\sigma, \rho).$$

This means that the probability that any operation (e.g., a measurement) performed on $\sigma$ can be distinguished from the same operation on $\rho$ is at most $\delta(\sigma, \rho)$.

*b*) Show that any purification of the state $\rho_{AB} = \frac{\mathbb{1}_A}{|\mathcal{H}_A|} \otimes \rho_B$ has the form

$$|\psi\rangle_{AA'BB'} = |\Psi\rangle_{AA'} \otimes |\psi\rangle_{BB'},$$

where $|\Psi\rangle_{AA'} = |\mathcal{H}_A|^{-\frac{1}{2}} \sum_i |i\rangle_A |i\rangle_{A'}$ is a maximally entangled state, and $|\psi\rangle_{BB'}$ is a purification of $\rho_B$.

*c*) Show that $1 - F(\rho, \sigma) \leq \delta(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}$.

*d*) Consider a state $\sigma_{AB}$ that is $\varepsilon$-distant from $\rho_{AB}$ according to the trace distance, i.e.

$$\delta\left(\sigma_{AB}, \frac{\mathbb{1}_A}{|\mathcal{H}_A|} \otimes \rho_B\right) \leq \varepsilon.$$

Find an upper bound for

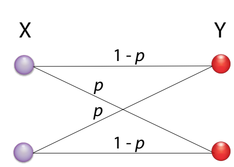
$$\delta\left(|\phi\rangle_{ABP}, |\Psi\rangle_{AA'} \otimes |\psi\rangle_{BB'}\right),$$

where $|\phi\rangle_{ABP}$ is a purification of $\sigma_{AB}$. You can take for instance $\mathcal{H}_P = \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$.

### Exercise 6.2 Classical channels as trace-preserving completely positive maps.

In this exercise we will see how to represent classsical channels as trace-preserving completely positive maps (TPCPMs).

*a*) Take the binary symmetric channel $\mathbf{p}$,

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Recall that we can represent the probability distributions on both ends of the channel as quantum states in a given basis: for instance, if $P_X(0) = q, P_X(1) = 1 - q$, we may express this as the 1-qubit mixed state $\rho_X = q\ |0\rangle\langle 0| + (1 - q)\ |1\rangle\langle 1|$.

What is the quantum state $\rho_Y$ that represents the final probability distribution $P_Y$ in the computational basis?

*b*) Now we want to represent the channel as a map

$$\begin{aligned}\mathcal{E}_{\mathbf{p}} : \mathcal{S}(\mathcal{H}_X) &\mapsto \mathcal{S}(\mathcal{H}_Y)\\ \rho_X &\to \rho_Y.\end{aligned}$$

An operator-sum representation (also called the Kraus-operator representation) of a CPTP map $\mathcal{E} : \mathcal{S}(\mathcal{H}_X) \to \mathcal{S}(\mathcal{H}_Y)$ is a decomposition $\{E_k\}_k$ of operators $E_k \in \mathrm{Hom}(\mathcal{H}_X, \mathcal{H}_Y)$, $\sum_k E_k {E_k}^\dagger = \mathbb{1}$, such that

$$\mathcal{E}(\rho_X) = \sum_k E_k \rho_X {E_k}^\dagger.$$

Find an operator-sum representation of $\mathcal{E}_{\mathbf{p}}$.

**Hint:** think of each operator $E_k = E_{xy}$ as the representation of the branch that maps input $x$ to output $y$.

*c*) Now we have a representation of the classical channel in terms of the evolution of a quantum state. What happens if the initial state $\rho_X$ is not diagonal in the computational basis?

*d*) Now consider an arbitrary classical channel $\mathbf{p}$ from an $n$-bit space $X$ to an $m$-bit space $Y$, defined by the conditional probabilities $\left\{P_{Y|X=x}(y)\right\}_{xy}$.

Express $\mathbf{p}$ as a map $\mathcal{E}_{\mathbf{p}} : \mathcal{S}(\mathcal{H}_X) \to \mathcal{S}(\mathcal{H}_Y)$ in the operator-sum representation.

### Exercise 6.3 TPCPMs as channels

Now we will go the other way around: we are given a TPCPM and will find a way of expressing it as a channel, and compute its capacity.
Consider two single-qubit Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and a TPCPM

$$\begin{aligned}\mathcal{E}_p : \mathcal{S}(\mathcal{H}_X) &\mapsto \mathcal{S}(\mathcal{H}_Y)\\ \rho &\to p\frac{\mathbb{1}}{2} + (1-p)\rho.\end{aligned}$$

*a*) Find an operator-sum representation for $\mathcal{E}_p$.

**Hint:** Remember that $\rho \in \mathcal{S}(\mathcal{H}_A)$ can be written in the Bloch sphere representation:

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}), \quad \vec{r} \in \mathbb{R}^3, \quad |\vec{r}| \leq 1, \quad \vec{r}\cdot\vec{\sigma} = r_x\sigma_x + r_y\sigma_y + r_z\sigma_z, \tag{1}$$

where $\sigma_x$, $\sigma_y$ and $\sigma_z$ are Pauli matrices. It may be useful to show that

$$\mathbb{1} = \frac{1}{2}(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z).$$

*b*) What happens to the Bloch radius $\vec{r}$ of the initial state when we apply $\mathcal{E}_p$? How can this be interpreted?

*c*) Now we will see what happens when we use this quantum channel to send classical information. We start with an arbitrary input probability distribution $P_X(0) = q, P_X(1) = 1-q$. We encode this distribution in a state $\rho_X = q\,|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|$. Now we send $\rho_X$ over the quantum channel, i.e., we let it evolve under $\mathcal{E}_{\mathbf{p}}$. Finally, we measure the output state, $\rho_Y = \mathcal{E}_{\mathbf{p}}(\rho_X)$ in the computational basis.

Compute the conditional probabilities $\left\{P_{Y|X=x}(y)\right\}_{xy}$.

*d*) Maximise the mutual information over $q$ to find the classical channel capacity of the depolarizing channel.

*e*) What happens to the channel capacity if we measure the final state in a different basis?