Sheet 6

Deadline: 14 November 2011

Exercise 1 [*Commutation relations of massless vector fields*]: The equal time commutation relations of the vector fields are

$$\begin{aligned} [A_\mu(t,\mathbf{x}), A_\nu(t,\mathbf{x}')] &= [\dot{A}_\mu(t,\mathbf{x}), \dot{A}_\nu(t,\mathbf{x}')] = 0 \\ [A_\mu(t,\mathbf{x}), \dot{A}_\nu(t,\mathbf{x}')] &= -ig_{\mu\nu}\, \delta^{(3)}(\mathbf{x}-\mathbf{x}') \ . \end{aligned} \tag{1}$$

Expand the fields in terms of modes as

$$A_\mu(t,\mathbf{x}) = \int d\tilde{k} \sum_{\lambda=0}^{3} \left[a^{(\lambda)}(k)\, \epsilon_\mu^{(\lambda)}(k)\, e^{-ik\cdot x} + a^{(\lambda)\dagger}(k)\, \epsilon_\mu^{(\lambda)}(k)\, e^{ik\cdot x} \right] \ , \tag{2}$$

where

$$d\tilde{k} = \frac{d^3k}{(2\pi)^3\, 2k_0} \ , \qquad k_0 = |\mathbf{k}| \ , \tag{3}$$

and the $\epsilon_\mu^{(\lambda)}(k)$ are polarisation vectors that satisfy

$$\sum_{\lambda=0}^{3} \frac{\epsilon_\mu^{(\lambda)}(k)\, \epsilon_\nu^{(\lambda)}(k)}{\epsilon^{(\lambda)}(k)\cdot \epsilon^{(\lambda)}(k)} = g_{\mu\nu} \ , \qquad \text{and} \qquad \epsilon^{(\lambda)}(k)\cdot\epsilon^{(\lambda')}(k) = g^{\lambda\lambda'} \ . \tag{4}$$

Show that the above equal time commutation relations (1) imply

$$\begin{aligned} [a^{(\lambda)}(k), a^{(\lambda')}(k')] &= [a^{(\lambda)\dagger}(k), a^{(\lambda')\dagger}(k')] = 0 \\ [a^{(\lambda)}(k), a^{(\lambda')\dagger}(k')] &= -g^{\lambda\lambda'} 2k^0\, (2\pi)^3\, \delta^{(3)}(\mathbf{k}-\mathbf{k}') \ . \end{aligned} \tag{5}$$

Exercise 2 [*Stückelberg theory*]:

(i) Show that the Stückelberg Langrangian

$$\mathcal{L} = -\frac{1}{4}F^2 + \frac{1}{2}\mu^2 A^2 - \frac{1}{2}\eta\, (\partial\cdot A)^2 \tag{6}$$

leads to the equations of motion

$$\left(\Box + \mu^2\right) A^\rho - (1-\eta)\, \partial^\rho\, (\partial\cdot A) = 0 \ . \tag{7}$$

(ii) Show that the mode expansion

$$\begin{aligned} A_\rho(x) \ &= \int \frac{d^3k}{2\,k^0\,(2\pi)^3} \sum_{\lambda=1}^{3} \left[a^{(\lambda)}(k)\, \epsilon_\rho^{(\lambda)}(k)\, e^{-ik\cdot x} + a^{(\lambda)\dagger}(k)\, \epsilon_\rho^{(\lambda)}(k)\, e^{ik\cdot x} \right]\Big|_{k^0=\sqrt{\mathbf{k}^2+\mu^2}} \\ &\quad + \int \frac{d^3k}{2\,k^0\,(2\pi)^3}\, \frac{k_\rho}{\mu}\, \left[a^{(0)}(k)\, e^{-ik\cdot x} + a^{(0)\dagger}(k)\, e^{ik\cdot x} \right]\Big|_{k^0=\sqrt{\mathbf{k}^2+m^2}} \ , \end{aligned}$$

solves the equation of motion (7), where $m^2 = \mu^2/\eta$ and the polarisation vectors $\epsilon^{(i)}(k)$, $i = 1, 2, 3$, are chosen so that

$$\epsilon^{(i)}(k) \cdot \epsilon^{(j)}(k) = -\delta^{ij} \ , \qquad \epsilon^{(i)}(k) \cdot k = 0 \ . \tag{8}$$

(iii) Using the form of the canonical commutation relations

$$[a^{(\lambda)}(k), a^{(\lambda')\dagger}(k')] = \delta_{\lambda\lambda'} \, (2\pi)^3 \, 2\sqrt{\mathbf{k}^2 + \mu^2} \, \delta^{(3)}(\mathbf{k} - \mathbf{k}') \qquad \lambda, \lambda' \in \{1, 2, 3\} \tag{9}$$

and

$$[a^{(0)}(k), a^{(0)\dagger}(k')] = -(2\pi)^3 \, 2\sqrt{\mathbf{k}^2 + m^2} \, \delta^{(3)}(\mathbf{k} - \mathbf{k}') \ , \tag{10}$$

calculate the Feynman propagator, and show that it takes the form

$$\langle 0| \mathcal{T}\Big(A_\rho(x)\, A_\nu(y)\Big)|0\rangle = -i \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot(x-y)} \left(\frac{g_{\rho\nu} - k_\rho k_\nu/\mu^2}{k^2 - \mu^2 + i\epsilon} + \frac{k_\rho k_\nu/\mu^2}{k^2 - m^2 + i\epsilon} \right) \ . \tag{11}$$

Hint: Derive the identities for the polarisation vectors from the electromagnetic case, eq. (4), with

$$\epsilon^{(0)}_\nu = \frac{k_\nu}{\mu} \ .$$

After you have calculated the time ordered expectation value from first principles, work backwards from the given answer to show that the two expressions agree.

(iv) Show that in the limit of $\eta \to 0$ with $\mu \neq 0$, (11) reproduces the propagator of the Proca theory,

$$\langle 0| \mathcal{T}\Big(A_\rho(x)\, A_\nu(y)\Big)|0\rangle = -i \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot(x-y)} \, \frac{g_{\rho\nu} - k_\rho k_\nu/\mu^2}{k^2 - \mu^2 + i\epsilon} - \frac{i}{\mu^2} \, \delta_{\rho 0} \, \delta_{\nu 0} \, \delta^{(4)}(x-y) \ . \tag{12}$$

(v) Show that in the limit $\mu \to 0$ with $\eta \neq 0$, (11) reproduces the propagator of the massless photon field with Feynman parameter η,

$$\langle 0| \mathcal{T}\Big(A_\rho(x)\, A_\nu(y)\Big)|0\rangle = -i \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot(x-y)} \, \frac{g_{\rho\nu} - (1 - \eta^{-1}) \frac{k_\rho k_\nu}{k^2}}{k^2 + i\epsilon} \ . \tag{13}$$