Sheet 2

Deadline: 17 October 2011

Exercise 1 [*Lorentz algebra*]:
Show that the matrices

$$(\mathcal{J}^{\mu\nu})_\alpha{}^\beta = i(\delta^\mu_\alpha g^{\nu\beta} - \delta^\nu_\alpha g^{\mu\beta})$$

define a representation of the Lorentz algebra, i.e. the Lie algebra of the Lorentz group; this means that they respect the relations

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i\,(g^{\nu\rho}\,\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\,\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\,\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\,\mathcal{J}^{\nu\rho}) \ . \tag{1}$$

(The corresponding representation is the usual irreducible 4-dimensional representation of the Lorentz group on 4-dimensional vectors in Minkowski space.)

Exercise 2 [*Construction of γ-matrices*]:
Suppose c_i, $i = 1, 2$ are two fermionic annihilation operators, with c_i^* the corresponding creation operators

$$\{c_i\,, c_j\} = \{c_i^*\,, c_j^*\} = \{c_1, c_2^*\} = \{c_2, c_1^*\} = 0\,, \quad \{c_1\,, c_1^*\} = \{c_2\,, c_2^*\} = 1\ . \tag{2}$$

These operators act on a fermionic Fock space $\mathcal{F}$ that is generated by the action of the creation operators from the vacuum Ω with $c_i\Omega = 0$, $i = 1, 2$.

(i) Show that the fermionic Fock space has dimension $\dim \mathcal{F} = 4$.
(ii) Define γ-operators via

$$\begin{aligned} \gamma^0 &= (c_1 + c_1^*) \qquad & \gamma^1 &= i(c_2 + c_2^*) \\ \gamma^2 &= (c_1 - c_1^*) \qquad & \gamma^3 &= (c_2 - c_2^*)\ . \end{aligned}$$

Using the anticommutation relations (2) show that these γ-operators satisfy the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}\ . \tag{3}$$

(iii) Since $\dim \mathcal{F} = 4$, these γ-operators define 4×4 matrices. Determine them explicitly and verify that the matrices satisfy indeed the Dirac algebra.

Exercise 3 [*γ-matrix representation of Lorentz algebra*]:
Suppose the γ^μ matrices satisfy the relations of the Dirac algebra (3). Show that the operators

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] \tag{4}$$

satisfy the relations of the Lorentz algebra (1), i.e.

$$[S^{\mu\nu}, S^{\rho\sigma}] = i \left(g^{\nu\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho} + g^{\mu\sigma} S^{\nu\rho} \right) \ .$$

Exercise 4 [*Chirality operator*]:
In 4 dimensions we can define the chirality operator

$$\gamma^5 = i\, \gamma^0\, \gamma^1\, \gamma^2\, \gamma^3 \ .$$

Using the relations of the Dirac algebra (3), show that the chirality operator satisfies

$$\{\gamma^5, \gamma^\mu\} = 0 \quad (\mu = 0, \ldots, 3) \ , \qquad \gamma^5\gamma^5 = \mathbf{1} \ .$$

Conclude that γ^5 commutes with the action of the operators (4) that were defined in Exercise 3,

$$[\gamma^5, S^{\mu\nu}] = 0 \ .$$

In particular, the Lorentz generators $S^{\mu\nu}$ therefore leave the eigenspaces of $\gamma^5 = \pm 1$ invariant. For the case of the usual 4-dimensional γ-matrices, show that the eigenspaces of $\gamma^5 = \pm 1$ are both 2- dimensional. Hence deduce that the corresponding 4-dimensional representation of the Lorentz algebra defined by $S^{\mu\nu}$ as in (4) decomposes as a direct sum of two 2-dimensional representations of the Lorentz algebra.