Sheet 11

Deadline: 19 December 2011

Exercise 1 [*Møller formula*]: In this exercise we shall derive the Møller formula for the differential cross-section for electron-electron scattering at leading order in the electron charge e.

Using the LSZ reduction formula for this process we can write the connected part T of the S-matrix $S = 1 + iT$ as

$$T = \langle p_1', p_2' | T | p_1, p_2 \rangle = (-iZ_2^{-1/2})^4 \int d^4x_1\, d^4x_2\, d^4x_1'\, d^4x_2'\, e^{i(p_1'\cdot x_1' + p_2'\cdot x_2' - p_1\cdot x_1 - p_2\cdot x_2)}$$
$$\times \bar{u}_{\epsilon_1'}(p_1')\,(i\, \overrightarrow{\not\partial}_{x_1'} - m)\, \bar{u}_{\epsilon_2'}(p_2')\,(i\, \overrightarrow{\not\partial}_{x_2'} - m)$$
$$\times \langle 0 | \mathcal{T}\left(\psi(x_1')\,\psi(x_2')\,\bar\psi(x_1)\,\bar\psi(x_2)\right) |0\rangle_c$$
$$\times (-i\, \overleftarrow{\not\partial}_{x_1} - m)\, u_{\epsilon_1}(p_1)\, (-i\, \overleftarrow{\not\partial}_{x_2} - m)\, u_{\epsilon_2}(p_2)\,,$$

where p_1, p_2 are the 4-momenta of the incoming particles, while p_1', p_2' are the 4-momenta of the outgoing particles. (To lowest order in e we only have tree diagrams that contribute as $Z_2^{-2}e^2$; thus we may set $Z_2 = 1$.) You should then proceed as follows:

(i) Rewrite T using the Fourier transform of the connected Green's function $G_c(p_1, p_2, p_1', p_2')$. Assume that $p_i \neq p_j'$ so that only the interacting part of the S-matrix contributes.

(ii) Use the Feynman rules for connected & amputated Green's functions in momentum space (see lectures) to obtain, to leading order in e

$$i\mathcal{M} = ie^2 \left(\frac{\bar{u}_{\epsilon_1'}(p_1')\gamma^\nu u_{\epsilon_1}(p_1)\bar{u}_{\epsilon_2'}(p_2')\gamma_\nu u_{\epsilon_2}(p_2)}{(p_1 - p_1')^2} - \frac{\bar{u}_{\epsilon_2'}(p_2')\gamma^\nu u_{\epsilon_1}(p_1)\bar{u}_{\epsilon_1'}(p_1')\gamma_\nu u_{\epsilon_2}(p_2)}{(p_1 - p_2')^2} \right), \quad (1)$$

where $\mathcal{M}$ is the matrix element for this process defined by

$$\langle p_1', p_2' | T | p_1, p_2 \rangle = (2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2) \cdot \mathcal{M}\ .$$

Notice that the two contributions in (1) come with different signs. This is to be expected since they are related by an exchange of identical fermions.

(iii) In the lecture we derived an expression for the total cross-section σ in the laboratory frame. Since $\sqrt{(p_1\cdot p_2)^2 - m^4} = m|\mathbf{p}_1|$ in the laboratory frame, the corresponding Lorentz invariant expression must be given by

$$d\sigma = \frac{\overline{|\mathcal{M}|^2}}{4\sqrt{(p_1\cdot p_2)^2 - m^4}} (2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2) \frac{d^3\mathbf{p}_1'}{(2\pi)^3 2E_1'} \frac{d^3\mathbf{p}_2'}{(2\pi)^3 2E_2'}\ , \quad (2)$$

where the bar over the squared matrix element $|\mathcal{M}|^2$ denotes spin averaging,

$$\overline{|\mathcal{M}|^2} = \frac{1}{4} \sum_{\epsilon_1, \epsilon_2, \epsilon_1', \epsilon_2'} |\mathcal{M}|^2\ . \quad (3)$$

Now go to the center of mass frame of the two incoming particles, i.e. to the frame for which

$$p_1 = (E, \mathbf{p}) \ , \qquad p_2 = (E, -\mathbf{p}) \ .$$

By integrating out the $\delta^{(4)}$-function show that (2) reduces to

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\overline{|\mathcal{M}|^2}}{64E^2(2\pi)^2} \ ,$$

where $d\Omega$ is the solid angle element corresponding to $\mathbf{p}'_1$, i.e. $d^2\mathbf{p}'_1 = |\mathbf{p}'_1|^2 \, d|\mathbf{p}'_1| \, d\Omega$.

Hints:

- First show $\sqrt{(p_1 \cdot p_2)^2 - m^4} = 2E\sqrt{E^2 - m^2}$ in the center of mass frame.
- Use the vectorial part of the $\delta^{(4)}$ function to integrate out $\mathbf{p}'_2$.
- Go to spherical coordinates for $\mathbf{p}'_1$ and integrate out $|\mathbf{p}'_1|$ using the remaining δ function, recalling that

$$\delta\left(f(x)\right) = \sum_{x_0} \frac{\delta(x - x_0)}{|f'(x_0)|} \ , \tag{4}$$

  where the function f only has simple zeros x_0, i.e. $f(x_0) = 0$ implies that $f'(x_0) \neq 0$; the sum in (4) runs then over all these (simple) zeros x_0.

(iv) Finally calculate the spin averaged matrix element (3), and thus derive the Møller formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2(2E^2 - m^2)^2}{4E^2(E^2 - m^2)^2} \left[\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta} + \frac{(E^2 - m^2)^2}{(2E^2 - m^2)^2}\left(1 + \frac{4}{\sin^2\theta}\right)\right] \ ,$$

where $\alpha = e^2/(4\pi)$ is the fine structure constant and θ the scattering angle of the two outgoing particles in the center of mass frame, i.e. $\theta = \measuredangle(\mathbf{p}, \mathbf{p}'_1) = \measuredangle(-\mathbf{p}, \mathbf{p}'_2) \in [0, \pi]$.

Hints:

- Using the symmetry in (1) you only need to compute $|\mathcal{M}_1|^2$ and $\mathcal{M}_1^\dagger \mathcal{M}_2$, where $i\mathcal{M} = i\mathcal{M}_1 - i\mathcal{M}_2$.
- Use the spin sums to express the products of spinors as traces over chains of γ-matrices.
- In order to compute the traces you may want to use the following identities

$$\begin{aligned}
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} \\
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \\
\gamma^\mu\gamma^\nu\gamma_\nu &= -2\gamma^\nu \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\nu &= 4g^{\nu\rho} \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\nu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu \ .
\end{aligned}$$

- Since the process is invariant under rotations around the beam axis the azimuthal angle is trivial. You can then choose without loss of generality to work in the x-z plane, where

$$\mathbf{p} = p\hat{\mathbf{z}}, \quad p'_1 = (E, p\sin\theta, 0, p\cos\theta), \quad p'_2 = (E, -p\sin\theta, 0, -p\cos\theta).$$