Sheet 10

Deadline: 12 December 2011

Exercise 1 [*Linear sigma model*]:
The interactions of pions at low energy can be described by a phenomenological model, called the *linear sigma model.* Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Phi^i(x)$, $i = 1, \ldots, N$ be a set of N fields, governed by the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}(\Pi^i)^2 + \frac{1}{2}(\nabla\Phi^i)^2 + V(\Phi^2) \right) \ , \tag{1}$$

where $\sum_i (\Phi^i)^2 = \vec{\Phi} \cdot \vec{\Phi}$, and

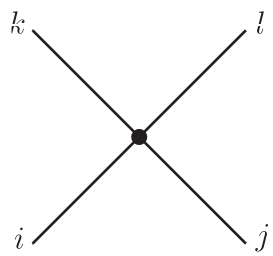
$$V(\Phi^2) = \frac{1}{2} m^2 (\vec{\Phi} \cdot \vec{\Phi}) + \frac{\lambda}{4} \big(\vec{\Phi} \cdot \vec{\Phi}\big)^2 \tag{2}$$

is a function symmetric under rotations of $\vec{\Phi}$. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H; hence, V is the field potential energy.

(i) Analyse the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes in terms of a perturbation series in the parameter λ. Show that the propagator is

$$\Phi^i(x)\Phi^j(y) = \delta^{ij} D_F(x - y) \ , \tag{3}$$

where D_F is the standard Klein-Gordon propagator for mass m, and that there is one type of vertex given by



$$= -2i\lambda \left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl} \right) .$$

(This is to say, the vertex between two Φ^1's and two Φ^2's has the value $(-2i\lambda)$; that between four Φ^1's has the value $(-6i\lambda)$.)

Compute, to leading order in λ, the differential cross section $\frac{d\sigma}{d\Omega}$ in the center-of-mass frame, for the scattering processes

$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \qquad \text{and} \qquad \Phi^1\Phi^1 \to \Phi^1\Phi^1 \tag{4}$$

as functions of the center-of-mass energy.

Hint: The formula for the differential cross section in the case where all particles have the same mass is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{CM}}^2} \ , \tag{5}$$

where CM indicates the center-of-mass frame. The invariant matrix element $\mathcal{M}$ is defined by

$$\langle p_1', p_2', \ldots |iT|\, p_1, p_2\rangle = (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum p_f\right) \cdot i\mathcal{M}(p_1, p_2 \to p_f) \ , \tag{6}$$

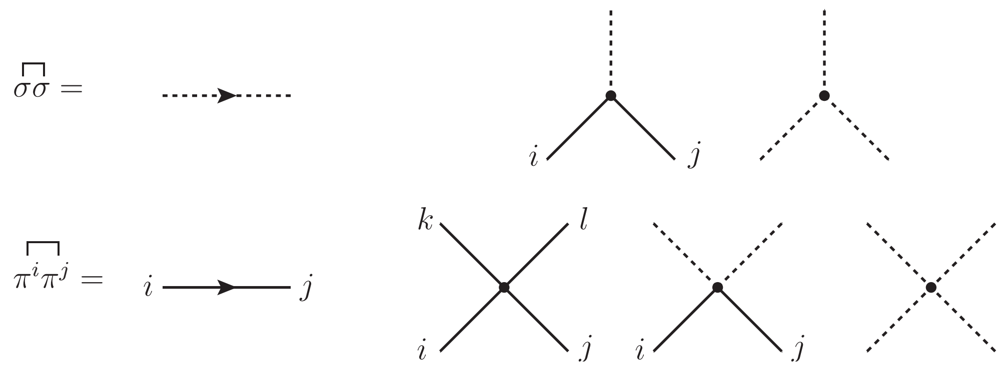
where T is the part of the S-matrix due interactions, i.e.

$$S = 1 + iT = \mathcal{T}\left(e^{-i\int d^4x\, H_{\text{int}}(x)}\right) \ . \tag{7}$$

(ii) Now consider the case $m^2 < 0$, introducing the parameter $m^2 = -\mu^2$. In this case V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i towards the minimum of V. By rotational invariance, we can consider this shift to be in the N^{th} direction. Thus we make the ansatz

$$\begin{aligned} \Phi^i(x) &= \pi^i(x) \ , \quad i = 1, \ldots, N-1 \\ \Phi^N(x) &= v + \sigma(x) \ , \end{aligned} \tag{8}$$

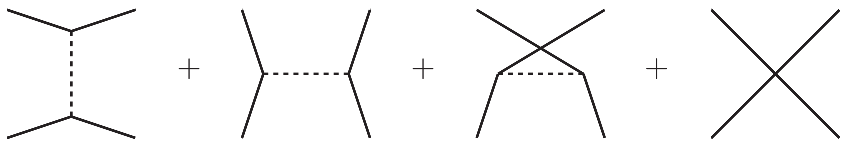
where v is a constant chosen so as to minimise V. (The notation π^i is meant to suggest a relation to the pion field, and should not be confused with the canonical momentum.) Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ *massless* pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$. Construct the Feynman rules by assigning values to the propagators and vertices



(iii) Compute the scattering amplitude for the process

$$\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^l(p_4) \tag{9}$$

to leading order in λ. There are now four Feynman diagrams that contribute



Show that, at threshold ($\vec{p}_i = 0$), these diagrams sum to *zero*.

Hint: It may be easiest to first consider the specific process $\pi^1\pi^1 \to \pi^2\pi^2$, for which only the first and fourth diagrams are nonzero, before tackling the general case.

Show that, in the special case $N = 2$ (1 species of pion), the terms at $\mathcal{O}(p^2)$ also cancel.

(iv) Add to V a symmetry-breaking term,

$$\Delta V = -a\,\Phi^N \ , \tag{10}$$

where a is a (small) constant. (In QCD, a term of this form is produced if the u and d quarks have the same non-vanishing mass.) Find the new value of v that minimises V, and work out the content of the theory about that point. Show that the pion acquires a mass with $m_\pi^2 \sim a$, and show that the pion scattering amplitude at threshold is now non-vanishing and also proportional to a.