General broken symmetries:

consider the case of a Lie group G broken spontaneously to a subgroup H

- Lagrangian is invariant under G:

$$\Psi_n(x) \to \sum_m g_{nm} \Psi_m(x) \qquad \text{with } g \in G$$

- all vacuum expectation values invariant under H:

$$\sum_m h_{nm} \langle \Psi_m(x) \rangle_{vac} = \langle \Psi_n(x) \rangle_{vac}$$

with $h \in H$

- Use real representation of G, H

Redefinition of the fields:

- introduce fields $\tilde{\Psi}_m(x)$, which correspond to fields with Goldstone modes removed
- recover full fields by

$$\Psi_n(x) = \sum_m \gamma_{nm}(x) \, \tilde{\Psi}_m(x)$$

transformations due to Goldstone modes

Conditions on $\tilde{\Psi}_n(x)$:

• orthogonality on Goldstone modes:

linear combinations of : $\sum_m [t^\alpha]_{nm} \langle \Psi_m(0) \rangle_{vac}$

generators of G

$$\Rightarrow \sum_{nm} \tilde{\Psi}_n(x) \, [t^\alpha]_{nm} \, \langle \Psi_m(0) \rangle_{vac} = 0$$

total number of conditions: $\dim G - \dim H$

$\Rightarrow$ need $(\dim G - \dim H)$ fields to parametrize $\gamma_{nm}(x)$

How to find $\gamma(x)$?

• consider

$$V_{\Psi(x)}(g \in G) = \sum_{nm} \Psi_n(x) g_{nm} \langle \Psi_m(x) \rangle_{vac}$$

(is a real function of g, bounded since G is compact)

$\Rightarrow$ $V_{\Psi(x)}(g)$ has maximum for $g = \gamma(x)$ in each space-time point x

• infinitesimal variation on g:

$$\delta g = i \sum_\alpha \varepsilon_\alpha \, g \, t^\alpha$$

$\varepsilon_\alpha \in \mathbb{R}$ infinitesimal

• extremal condition on $V_{\varphi(x)}(g)$

$$0 = \delta V_{\varphi(x)}(\gamma(x)) = i\, \varepsilon_\alpha \sum_{mnl} \varphi_n(x) \gamma_{nl}(x) t^\alpha_{lm} \langle \varphi_m(0)\rangle$$

$$= i \sum_\alpha \varepsilon_\alpha \sum_{mnl} [\gamma^{-1}(x)]_{ln} \varphi_n(x) t^\alpha_{lm} \langle \varphi_m(0)\rangle_{vac}$$

must be fulfilled for all ε_α

$\Rightarrow \tilde{\varphi}(x) = \gamma^{-1}(x)\varphi(x)$

is orthogonal on all Goldstone modes.

• $\gamma(x)$ not unique, since vacuum is invariant under H:

$$V_{\varphi(x)}(g) = V_{\varphi(x)}(g\, h) \qquad \text{for } h \in H$$

right multiplication.

such that:

$$\delta V_{\varphi(x)}(\gamma(x)) = 0 \Rightarrow \delta V_{\varphi(x)}(\gamma(x) h) = 0$$

and

$\tilde{\varphi} = h^{-1}\gamma^{-1}\varphi$ and $\tilde{\varphi} = \gamma^{-1}\varphi$

both orthogonal on Goldstone modes

$\Rightarrow \quad \gamma_1 = \gamma_2 h$ and γ_2 equivalent

Sort elements of G into disjoint equivalence classes, each consisting of elements differing only by right multiplication with $h \in H$:

right cosets of G w.r.t. H : G/H

- <u>need to find only a single element from each right coset G/H</u>

- Generators of H: t_i

with $\sum_m (t_i)_{\mu m} (\gamma_m)_{\nu a} = 0$

form a sub-algebra

$$[t_i, t_j] = i \sum_k C_{ijk} t_k$$

- Generators of G: t_i and remaining x_a

$(C_{ija} = 0$ and $C_{iaj} = 0)$

such that: $\quad [t_i, x_a] = i \sum_b C_{iab} x_b$

and: $[x_a, x_b] = i \sum_i C_{abi} t_i + i \sum_c C_{abc} x_c$

(Cartan decomposition of Lie algebra)

any element of G can be represented as

$$g = \exp\left(i \sum_a \xi_a x_a\right) \exp\left(i \sum_i \Theta_i t_i\right)$$

with $\xi_a, \Theta_i \in \mathbb{R}$

$\Rightarrow$ can standarize

$$\gamma(x) = \exp\left(i \sum_a \xi_a(x)\, x_a\right)$$

(eliminates coset - ambiguity)

$\xi_a(x)$: Goldstone boson fields

$$\gamma(x) = \gamma(\xi(x))$$

Replace $\psi(x)$ in $\mathcal{L}$

$$\partial_\mu \psi(x) = \gamma(x) \left[\partial_\mu \tilde\psi(x) + \left(\gamma^{-1}(x)\, \partial_\mu \gamma(x)\right) \tilde\psi(x)\right]$$

$\Rightarrow$ dependence on Goldstone fields only through $\gamma^{-1}(x)\, \partial_\mu \gamma(x)$ $\rightarrow$ derivative coupling

no-mass terms

expand $\gamma^{-1}(x)\partial_\mu \gamma(x)$ in group generators

$$\gamma^{-1}(x)\, \partial_\mu \gamma(x) = i \sum_a x_a D_{a\mu}(x) + i \sum_i t_i E_{i\mu}(x)$$

with
$$D_{a\mu}(x) = \sum_b D_{ab}(\xi(x))\, \partial_\mu \xi_b(x)$$
$$E_{i\mu}(x) = \sum_b E_{ib}(\xi(x))\, \partial_\mu \xi_b(x)$$

Transformation behavior under $\mathcal{G}$:

$$\psi(x) \longrightarrow \psi'(x) = g\, \psi(x)$$
$$= g\, \gamma(\xi(x))\, \tilde{\psi}(x)$$

$g\,(\gamma(\xi(x))) \in \mathcal{G} \Rightarrow$ there exists a $\gamma(\xi'(x))$ with

$$g\, \gamma(\xi(x)) = \underbrace{\gamma(\xi'(x))\, h(\xi(x), g)}$$

general form of element of $\mathcal{G}$

$$\Rightarrow \psi'(x) = \gamma(\xi'(x))\, \tilde{\psi}'(x)$$

with $\tilde{\psi}'(x) = h(\xi(x), g)\, \tilde{\psi}(x)$

- determines transformation of $\tilde{\psi}, \xi$ (complicated for general g)
- Special case: $g = h \in H$, with $R^h(h)$: linear rep. of h

$$h\, \gamma(\xi)\, h^{-1} = \gamma(R^h(h)\, \xi)$$

from commutation relations of x, t:

$$h\, x_b\, h^{-1} = \sum_a R^h_{ab}(h)\, x_a$$

follow

transformations of ξ and $\tilde{\psi}$:

$$\xi'_a(x) = \sum_b R^L_{ab}(h)\,\xi_b(x)$$

$$\tilde{\psi}'_n(x) = \sum_m h_{nm}\,\tilde{\psi}_m(x)$$

- derive transformation rules for $D_{a\mu}(x)$, $E_{i\mu}(x$

use: $\gamma^{-1}(\xi(x))\,\partial_\mu\gamma(\xi(x))$ invariant

$$= [g\gamma(\xi(x))]^{-1}\,\partial_\mu(g\gamma(\xi(x)))$$

$$= h^{-1}(\xi(x),g)\,\gamma^{-1}(\xi'(x))\,\partial_\mu[\gamma(\xi'(x))\,h(\xi(x),g)]$$

$$= h^{-1}(\xi(x),g)\,(\gamma^{-1}(\xi'(x))\,\partial_\mu\gamma(\xi'(x)))\,h(\xi(x),g)$$

$$+\,h^{-1}(\xi(x),g)\,\partial_\mu h(\xi(x),g)$$

such that:

$$\gamma^{-1}(\xi'(x))\,\partial_\mu\gamma(\xi'(x))$$

$$= h(\xi(x),g)\,(\gamma^{-1}(\xi(x))\,\partial_\mu\gamma(\xi(x)))\,h^{-1}(\xi(x),g)$$

$$-\,[\partial_\mu h(\xi(x),g)]\,h^{-1}(\xi(x),g)$$

comparison of coefficients yields:

$$\sum_a x_a D'_{a\mu} = h(\xi,g)\left(\sum_a x_a D_{a\mu}\right)h^{-1}(\xi,g)$$

$$\sum_i t_i E'_{i\mu} = h(\xi,g)\left(\sum_i t_i E_{i\mu}\right)h^{-1}(\xi,g) + i\left(\partial_\mu h(\xi,g)\right)h^{-1}(\xi,g)$$

$$\Rightarrow \quad D'_{a\mu}(x) = \sum_b R^L_{ab}(h(\xi),g)\, D_{b\mu}$$

transforms according to linear representation R^L

$$E'_{i\mu}(x) = \sum_j R_{ij}\left(h(\xi(x),g)\right)E_{j\mu}(x) - \sum_b H_{ib}\left(h(\xi(x),g)\right)\partial_\mu \xi_b(x)$$

with $\quad h\, t_j\, h^{-1} = \sum_i R_{ij}(h)\, t_i$

R not necessarily linear

and $\quad \partial_\mu h(\xi(x),g)\, h^{-1}(\xi(x),g) = i\sum_{ib} H_{ib}(\xi(x))\, t_i\, \partial_\mu \xi_b(x)$

$\rightarrow$ $E_{i\mu}(x)$ transforms like gauge field introduce covariant derivative:

$$D_\mu \tilde{\Psi}(x) = \partial_\mu \tilde{\Psi}(x) + i \sum_i t_i E_{i\mu}(x) \tilde{\Psi}(x)$$

using transformation under G

$$\partial_\mu \tilde{\Psi}'(x) = h(\xi(x), g)\left[\partial_\mu \tilde{\Psi}(x) + h^{-1}(\xi(x), g)\, \partial_\mu h(\xi(x), g)\, \tilde{\Psi}(x)\right]$$

yields

$$\left[D_\mu \tilde{\Psi}(x)\right]' = h(\xi(x), g)\, D_\mu \tilde{\Psi}(x)$$

$\Rightarrow$ any Lagrangian which is invariant under H (reduced symmetry of broken theory) and contains only

$$\underbrace{\tilde{\Psi},\ D_\mu \tilde{\Psi}}_{\text{"heavy fields"}},\ \underbrace{D_{\alpha\mu}(\xi)}_{\text{Goldstone fields: "light"}}$$

is also invariant under the larger group G.