2. Chiral symmetry in strong interactions

Lit: Peskin/Schroeder: Quantum Field Theory, Ch. 19

Yndurain: Theory of Quark and Gluon Interactions, Ch. 7

Donoghue, Golowich, Holstein: Dynamics of the Standard Model, Ch. 6

2.1. Chiral symmetry of QCD

Fermionic part of QCD Lagrangian: (u,d quarks only)

$$\mathcal{L} = \bar{u}\, i\not{D} u + \bar{d}\, i\not{D} d + m_u \bar{u}u + m_d \bar{d}d$$

$$= \bar{u}_L i\not{D} u_L + \bar{d}_L i\not{D} d_L + \bar{u}_R i\not{D} u_R + \bar{d}_R i\not{D} d_R$$

$$+ m_u (\bar{u}_R u_L + \bar{u}_L u_R) + m_d (\bar{d}_R d_L + \bar{d}_L d_R)$$

Symmetries for $m_{u,d} \to 0$

a) vector: • $SU(2)$ isospin: (violated by mass terms)

$$Q = \begin{pmatrix} u \\ d \end{pmatrix} \longrightarrow Q' = \mathcal{U}\, Q \qquad \mathcal{U} \in SU(2)$$

$$\Rightarrow j^{\mu a} = \bar{Q}\gamma^\mu \tau^a Q$$

• $U(1)$ baryon number (not violated by mass terms)

$$Q \to Q' = e^{i\theta} Q \qquad \Rightarrow j^\mu = \bar{Q}\gamma^\mu Q$$

b) axialvector: - $SU(2)$ isospin

$$Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \to Q'_L = \mathcal{U}_L Q_L$$

$$Q_R = \begin{pmatrix} u \\ d \end{pmatrix}_R \to Q'_R = \mathcal{U}_R Q_R \qquad \Rightarrow j_5^{\mu a} = \bar{Q}\gamma^\mu \gamma_5 \tau^a Q$$

• $U(1)$ baryon number

$$Q_L \to Q'_L = e^{i\theta_L} Q_L, \quad Q_R \to Q'_R = e^{i\theta_R} Q_R \Rightarrow j_5^\mu = \bar{Q}\gamma^\mu \gamma_5 Q$$

vector symmetries obtained for

$$U_L = U_R \; ; \; \Theta_L = \Theta_R$$

symmetry group $\quad SU(2)_L \otimes SU(2)_R \otimes U(1)_L \otimes U(1)_R$

or, alternatively

$$SU(2)_V \otimes SU(2)_A \otimes U(1)_V \otimes U(1)_A$$

- vector symmetries observed in strong interaction
  - isospin symmetry
  - baryon number conservation
- hypothesis: axial vector symmetries spontaneously broken (Nambu, Jona-Lasinio) by $Q\bar{Q}$ condensate in QCD vacuum

vacuum state: spinless, no ~~momentum~~

scalar: $\langle 0 | \bar{Q} Q | 0 \rangle$

$$= \langle 0 | \bar{Q}_L Q_R + \bar{Q}_R Q_L | 0 \rangle \neq 0$$

- has axial charge (not invariant under U_L, U_R separately)
- yields effective mass term for u,d quarks, even for $m_{u,d} = 0$
- breaks $SU(2)_L \otimes SU(2)_R \otimes U(1)_L \otimes U(1)_R$

$$\rightarrow SU(2)_V \otimes U(1)_V$$

- 4 broken generators

  $\Rightarrow$ 4 Goldstone bosons $\quad SU(2): \pi^a$

  (massless, pseudoscalar) $\quad U(1): \eta'$

Pions: · light pseudoscalars (would be exactly massless for $m_{u,d}=0$)

· annihilated by $j_5^{\mu a}$:

$$\langle 0 | j_5^{\mu a}(x) | \pi^b(p)\rangle = -i p^\mu f_\pi \delta^{ab} e^{-ip\cdot x}$$

$j_5^{\mu a}$  π^b $\qquad f_\pi$: pion decay constant ($= 93$ MeV)

· pion masses

compute

$$\langle 0 | \partial_\mu j_5^{\mu a}(x) | \pi^b(p)\rangle = -\underbrace{p^2}_{p^2 = m_\pi^2} f_\pi \delta^{ab} e^{-ipx}$$

- for $m_{u,d}=0$: $\partial_\mu j_5^{\mu a} = 0 \quad \Rightarrow m_\pi^2 = 0$

  (triplet current is not anomalous)

- for $m_{u,d} \neq 0$ $\quad M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$

$$i \not{D} Q = M Q \quad , \quad -i \bar{Q} \overleftarrow{\not{D}} = \bar{Q} M$$

such that: $\partial_\mu j_5^{\mu a} = i \bar{Q} \{M, \tau^a\} Q$

$$\langle 0 | \partial_\mu j_5^{\mu a}(x) | \pi^b(p)\rangle = \langle 0 | i \bar{Q}\, \mathrm{tr}(\{M, \tau^a\} \tau^b) Q | 0 \rangle f_\pi$$

with $\mathrm{tr}\left(\{M,\tau^a\}\tau^b\right) = \frac{1}{2}\delta^{ab}\,(m_u+m_d)$

$$m_\pi^2 = -\frac{m_u+m_d}{f_\pi^2}\,\langle 0|\bar{Q}Q|0\rangle$$

in numbers $m_u+m_d \simeq 10\,\mathrm{MeV}$, $m_\pi \simeq 140\,\mathrm{MeV}$

$$\Rightarrow \langle 0|\bar{Q}Q|0\rangle = -(\cancel{420\,\mathrm{MeV}}\ 260\,\mathrm{MeV})^3$$

- Leptonic decay of charged pions:

Axial vector current:

$$j_5^{\mu a} = \bar{Q}\gamma^\mu\gamma_5\tau^a Q = \bar{u}\gamma^\mu\gamma^5 d + \bar{d}\gamma^\mu\gamma^5 u + \bar{u}\gamma^\mu\gamma^5 u - \bar{d}\gamma^\mu\gamma^5 d$$

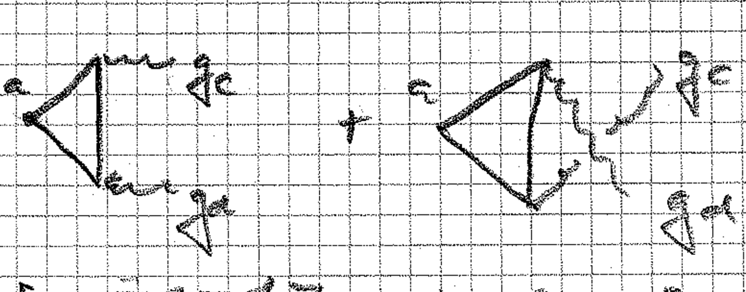
is probed by weak decays

$$\mathcal{L}^{\mathrm{Fermi}} = \frac{G_F}{\sqrt{2}}\,\bar{\mu}\gamma_\lambda(1-\gamma_5)\nu_\mu\,\bar{u}\gamma^\lambda(1-\gamma_5)d + \ldots$$

$\rightarrow$ exercise

- Anomalies of chiral currents

QCD:
a) Isotriplett $\partial_\mu j^{\mu 5 a} = -\frac{g^2}{16\pi^2}\varepsilon^{\alpha\beta\mu\nu}F^c_{\alpha\beta}F^d_{\mu\nu}\,\mathrm{tr}\left[\tau^a T^c T^d\right]$



$$\mathrm{tr}\left[\tau^a T^c T^d\right] = \mathrm{tr}\,\tau^a\;\mathrm{tr}\left[T^c T^d\right] = 0$$

$\Rightarrow j^{\mu 5 a}$ is anomaly-free in QCD
(conserved current for $m_{u,d} = 0$)

b) isosinglet: $\partial_\mu j^{\mu 5} = -\frac{g^2 n_f}{32\pi^2} \varepsilon^{\alpha\beta\mu\nu} F^c_{\alpha\beta} F^c_{\mu\nu} \neq 0$

$\Rightarrow j^{\mu 5}$ has anomaly in QCD
(not a conserved current even for $m_{u,d} = 0$)

$\rightarrow$ anomalous contribution to η' mass

$m_{\eta'} = 960\ MeV \gg m_\pi$

QED

$$\partial_\mu j_5^{\mu a} = -\frac{e^2}{16\pi^2} \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}\ \mathrm{tr}(\tau^a Q^2) \cdot N_c$$

with $Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \in SU(2)$

$$\mathrm{tr}(\tau^a Q^2) = \frac{1}{2}\delta^{a3}\ \mathrm{tr}\begin{pmatrix} \frac{4}{9} & 0 \\ 0 & -\frac{1}{9} \end{pmatrix} = \frac{1}{6}\delta^{a3}$$

$$\Rightarrow \partial_\mu j_5^{\mu 3} = -\frac{e^2}{32\pi^2} \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}$$

- anomaly in $j_5^{\mu 3}$ (π^0 decay) in QED mediates $\pi^0 \to \gamma\gamma$ decay axial current matrix element into two photon state

$$\langle \gamma(p)\gamma(k) \,|\, j_5^{\mu 3}(q) \,|\, 0\rangle = \varepsilon_\nu^* \varepsilon_\lambda^* M^{\mu\nu\lambda}(p,k)$$

using $p_\nu M^{\mu\nu\lambda} = 0 \;;\; k_\lambda M^{\mu\nu\lambda} = 0$

and $M^{\mu\nu\lambda}(p,k) = M^{\mu\lambda\nu}(k,p)$

yields general tensor decomposition:

$$M^{\mu\nu\lambda} = q^\mu \varepsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta M_1$$
$$+ (\varepsilon^{\mu\nu\alpha\beta} k^\lambda - \varepsilon^{\mu\lambda\alpha\beta} p^\nu) k_\alpha p_\beta M_2$$
$$+ [\varepsilon^{\mu\nu\alpha\beta} p^\lambda - \varepsilon^{\mu\lambda\alpha\beta} k^\nu) k_\alpha p_\beta - \varepsilon^{\mu\nu\lambda\sigma}(p-k)_\sigma \, p\cdot k]$$
$$M_3$$

$$i q_\mu M^{\mu\nu\lambda} = i q^2 \varepsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta M_1 - i \varepsilon^{\mu\nu\lambda\sigma} q_\mu (p-k)_\sigma \, p\cdot k \, M_3$$

$$= i q^2 \varepsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta (M_1 + M_3)$$

$$= -\frac{e^2}{4\pi^2} \varepsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta$$

the anomaly (must arise from $\frac{1}{q^2}$ contribution to M_1 or M_3)

parametrize $\pi^0 \to 2\gamma$ amplitude as:

$$i\mathcal{M}(\pi^0 \to 2\gamma) = i A \varepsilon_\nu^* \varepsilon_\lambda^* \varepsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta$$

contributes to $M^{\mu\nu\lambda}$ as $(i q^\mu f_\pi) \frac{i}{q^2} (i A \varepsilon_\nu^* \varepsilon_\lambda^*) \varepsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta$

Such that: $M_1 = -\frac{i}{q^2} f_\pi A$ + regular in q^2

$$\Rightarrow A = \frac{e^2}{4\pi^2} \frac{1}{f_\pi}$$

in $\pi^0 \to \gamma\gamma$ width

$$\Gamma(\pi^0 \to 2\gamma) = \frac{1}{2\pi} \frac{1}{8\pi} \frac{1}{2} \sum_{\lambda_\gamma} |M(\pi^0 \to 2\gamma)|^2$$

$$= \frac{1}{32\pi m_\pi} A^2 \, 2(p\cdot k)^2 = A^2 \frac{m_\pi^3}{64\pi}$$

$$= \frac{\alpha^2}{64\pi^3} \frac{m_\pi^3}{f_\pi^2}$$

$\Rightarrow$ direct measure of anomaly coefficient

2.2. σ-model

search: effective lagrangian to describe hadron interactions at low energies, with chiral and isospin symmetry

Starting point: linear σ-model of πN-interaction

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix} \qquad \vec{\pi} = \begin{pmatrix} \pi^1 \\ \pi^2 \\ \pi^3 \end{pmatrix} \qquad \sigma \qquad (m_p = m_n)$$

(SU(2) fund.) (SU(2) adj.) (SU(2) singlet)

$$\mathcal{L}_\sigma = \bar\psi i\partial\!\!\!/ \psi + \frac{1}{2}(\partial_\mu \vec\pi)(\partial^\mu \vec\pi) + \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma)$$

$$- g\bar\psi(\sigma - i\vec\tau\cdot\vec\pi\gamma_5)\psi + \underbrace{\frac{\mu^2}{2}(\sigma^2+\vec\pi^2) - \frac{\lambda}{4}(\sigma^2+\vec\pi^2)^2}$$

for $\mu^2>0$: SSB ; $\mu^2 = \lambda v^2$ rewrite

as: $-\frac{\lambda}{4}(\sigma^2+\pi^2-v^2)^2$

introduce : $\Sigma = \sigma \mathbb{1} - i\vec\tau\cdot\vec\pi$ $\qquad (\sigma^2+\pi^2) = \frac{1}{2}\mathrm{tr}(\Sigma^+\Sigma)$

$$\mathcal{L}_\sigma = \frac{1}{4}\mathrm{tr}(\partial_\mu\Sigma\partial^\mu\Sigma^+) - \frac{\lambda}{16}\left(\mathrm{tr}(\Sigma^+\Sigma) - 2v^2\right)^2$$

$$+ \bar\psi_L i\partial\!\!\!/ \psi_L + \bar\psi_R i\partial\!\!\!/ \psi_R - g\bar\psi_R\Sigma\psi_L - g\bar\psi_L\Sigma^+\psi_R$$

is invariant under $SU(2)_L \otimes SU(2)_R$

$$\psi_L \to g_L\psi_L \qquad \psi_R \to g_R\psi_R \qquad \Sigma \to g_R\Sigma g_L^{-1}$$

vacuum state :

$v^2 < 0$: $\Sigma = 0$ invariant

$v^2 > 0$: $\sigma = v$; $\vec\pi = 0$ not invariant

a) linear representation of $\mathcal{L}_\sigma$:

$\sigma = v + \tilde{\sigma}$, π : Goldstone bosons

$$\mathcal{L}_\sigma = \frac{1}{2}\left(\partial_\mu \tilde{\sigma}\,\partial^\mu \tilde{\sigma} - 2\mu^2 \tilde{\sigma}^2\right) + \frac{1}{2}\partial_\mu \vec{\pi}\cdot\partial^\mu \vec{\pi} - \lambda v\,\tilde{\sigma}\left(\tilde{\sigma}^2 + \vec{\pi}^2\right)$$

$$- \frac{\lambda}{4}\left(\tilde{\sigma}^2 + \vec{\pi}^2\right)^2 + \bar{\psi}\left(i\not{\partial} - g v\right)\psi - g\,\bar{\psi}\left(\tilde{\sigma} - i\vec{\tau}\cdot\vec{\pi}\gamma_5\right)\psi$$

$\rightarrow$ nucleon mass $g v$; $m_\sigma^2 = 2\mu^2$

b) Non-linear representation of $\mathcal{L}_\sigma$

$U(\varphi) = \exp\left(i\frac{\vec{\tau}\cdot\vec{\varphi}}{v}\right)$ $\quad \Sigma = (v+S)\,U(\varphi)$

$\vec{\varphi} = \vec{\pi} + \ldots$ $\quad \Psi_L = u\psi_L \quad \Psi_R = u^+\psi_R$

with $S^+ = S \quad u^+ = u^{-1} \quad \det u = 1 \, , \; u^2 = U$

yields:

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu S)^2 - 2\mu^2 S^2\right] + \frac{(v+S)^2}{4}\,\mathrm{tr}\left(\partial_\mu U\,\partial^\mu U^+\right)$$

$$- \lambda v S^3 - \frac{\lambda}{4}S^4 + \bar{\Psi}\,i\not{\partial}\Psi - g(v+S)\left(\bar{\Psi}_L U \Psi_R + \bar{\Psi}_R U^+ \Psi_L\right)$$

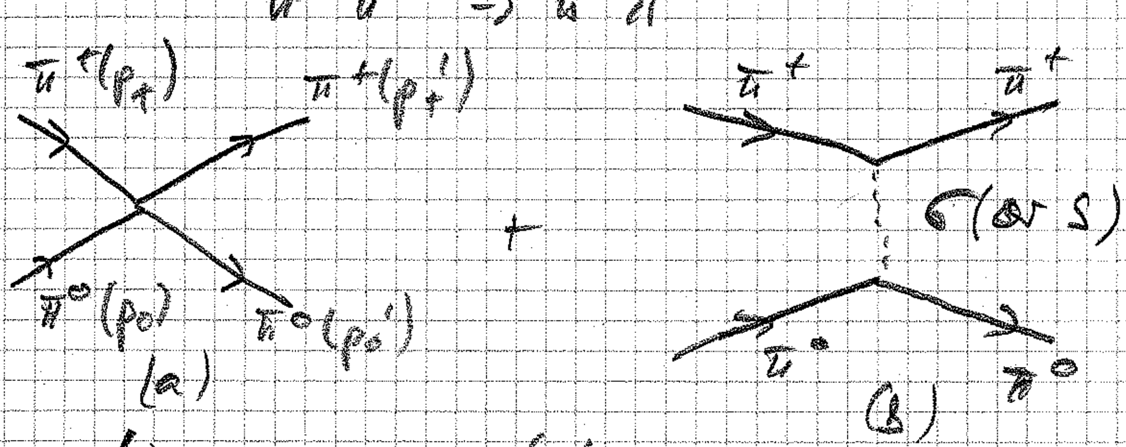
with $U \xrightarrow[SU(2)_L \otimes SU(2)_R]{} g_R\, U\, g_L^{-1}$

$$S \rightarrow S$$

only derivative couplings of Goldstone bosons (π), as expected.

Example: pion scattering

$$\pi^+\pi^0 \to \pi^+\pi^0$$

$M =$ 

a) in linear representation:

$$q = p_+' - p_+ = p_0' - p_0$$

$$\mathcal{L}_{int} = -\frac{\lambda}{4}\left(\vec{\pi}^2\right)^2 - \lambda v \sigma \vec{\pi}^2$$

such that

$$M = -2i\lambda + (-2i\lambda v)^2 \frac{i}{q^2 - m_\sigma^2}$$

expand in powers of momenta

$$= -2i\lambda\left[1 + \frac{2\lambda v^2}{q^2 - 2\lambda v^2}\right] = \frac{iq^2}{v^2} + O\left(\frac{q^4}{v^4}\right)$$

(M vanishes at $q^2 = 0$, although no momentum-dependent couplings)

b) in non-linear representation

$$\mathcal{L}_{int} = \frac{(v+S)^2}{4}\,\mathrm{tr}\left(\partial_\mu U \partial^\mu U^+\right) = \frac{1}{6v^2}\left[(\vec{\varphi}\cdot\partial_\mu\vec{\varphi})^2 - \vec{\varphi}^2(\partial_\mu\vec{\varphi}\cdot\partial_\mu\vec{\varphi})\right] + S\text{-terms}$$

expand in powers of momenta

- diagram (b) is p^4
- diagram (a): $M = \dfrac{i(p_+' - p_+)^2}{v^2} = \dfrac{iq^2}{v^2}$

$\Rightarrow$ scattering matrix elements are independent on representation

Haag theorem: same experimental observables obtained from

$$\mathcal{L}(\varphi) \quad \text{and} \quad \mathcal{L}(\chi f(\chi))$$

provided
- $\varphi = \chi f(\chi)$, $f(0) = 1$
- φ and χ have same free-field behaviour

2.3 Chiral perturbation theory

Search relation between quark-level (ew level) and hadron level (strong) interactions at low energies.

starting point: massless QCD lagrangian with N_f quark flavours

$$\mathcal{L}_{QCD} = \bar{q}\, i \not{D} q - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \qquad q = \begin{pmatrix} u \\ d \\ (s) \end{pmatrix}$$

with global symmetry

$$SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes U(1)_A$$