Review of earlier ~~courses~~ lectures

* Effective action
  - Physical vev's for field operators minimize it
$$\frac{\delta \Gamma}{\delta \langle \phi \rangle} = 0.$$
  - symmetry transformation
$$\langle \phi \rangle \to \langle \phi \rangle + \epsilon \langle F[\phi, \partial] \rangle$$
    keeps it invariant
    if $\phi \to \phi + \epsilon F[\phi]$
    * remember $\langle F[\phi] \rangle \neq F[\langle \phi \rangle]$ in general
    * continuous linear symmetry transformation of the action, are also symmetry transformation of the effective action

* Broken symmetries (spontaneously)
  - Symmetries of the effective action do not leave the vacuum state invariant
  - Degeneracy: $\langle \phi \rangle$ & $\underset{\text{symmetry transform}}{\underline{L}} \langle \phi \rangle$ = correspond

    to degenerate vacuum states characterized by different vev's of the field operator
  - Orthonormal ~~states~~ vacuum states are stable. Once in one of them, a tunneling through infinite space is required to move to another.

* Goldstone Theorem: SSB $\Rightarrow$ massless particles.

Proof a la Weinberg of Goldstone theorem

Consider an action which is invariant under a continuous transformation. From Nöther's theorem we find a conserved current

$$J^\mu(\vec{x},t)$$

with

$$\partial_\mu J^\mu = 0 \ .$$

The quantity $Q = \int d^3\vec{x}\, J^0(\vec{x},t)$ is conserved (independent of time). Under the symmetry transformation, quantum fields transform as

$$[Q, \varphi_n] = i\delta\varphi_n ,$$

where ϕ ~~and~~ Q are now operators.
For a continuous symmetry transformation

$$\epsilon\,\delta\phi_n = i\epsilon T_{nm}\,\phi_m$$

[The classical field transforms as $\varphi \to \varphi + \epsilon\delta\varphi$.]

To study SSB of the global symmetry, we start with the average value of the commutator

$$\frac{\langle 0|\,[J^\mu(y), \varphi_n(x)]\,|0\rangle}{\langle 0|0\rangle} \equiv \langle [J^\mu(y), \varphi_n(x)] \rangle$$

in the vacuum.

we have

$$\langle [J^\mu(y), \varphi_n(x)] \rangle = \langle J^\mu(y)\varphi_n(x) - \varphi_n(x) J^\mu(y) \rangle$$

$$= \sum_N \langle 0| J^\mu(y) |N\rangle \langle N| \phi_n(x) |0\rangle$$

$$- \langle 0| \phi_n(x) |N\rangle \langle N| J^\mu(y) |0\rangle . \qquad (\text{Eq. } 2)$$

We now use that

$$O(x) = e^{-i\hat{P}\cdot x}\, \hat{O}(0)\, e^{i\hat{P}\cdot x} .$$

[representation of translation symmetry]

and we have that, if $\hat{P}|\alpha\rangle = P_\alpha |\alpha\rangle$

& $\hat{P}|\beta\rangle = P_\beta |\beta\rangle$

then

$$\langle \alpha | \hat{O}(x) |\beta\rangle = e^{i(P_\alpha - P_\beta)\cdot x} \langle \alpha | \hat{O}(0) |\beta\rangle$$

(Eq. 3)

Therefore,

$$\langle [\;\;] \rangle = \sum_N \Big\{ \langle 0| J^\mu(0) |N\rangle \langle N| \phi_n(0) |0\rangle e^{-iP_N(x-y)}$$

$$+ \langle N| J^\mu(0) |0\rangle \langle 0| \phi_n(0) |N\rangle e^{+iP_N(x-y)} \Big\}$$

$$= \sum_N \Big\{ \Big[A^\mu(N, J^\mu)\, e^{-iP_N(x-y)} + \text{c.c} \Big]$$

We write,

$$\langle [\ ] \rangle = \int d^4p \sum_N \delta(p - p_N) \left\{ \langle 0 | J^\mu(0) | N \rangle \langle N | \phi_n(0) | 0 \rangle \right.$$

$$\left. e^{-ip\cdot(x-y)} + c.c \right\}$$

If we perform the sum over the states $|N\rangle$ we will obtain a result which depends only on the momentum p^μ. We will then have.

$$\sum_N \langle 0 | J^\mu(0) | N \rangle \langle N | \phi_n(0) | 0 \rangle \delta(p - p_N) =$$

$$= p^\mu \rho_n(p^2) \text{ . and thus :}$$

$$\langle 0 | [J^\mu(y), \phi_n(x)] | 0 \rangle = \int d^4p \ \theta(p^0) \left[p^\mu e^{-i(x-y)\cdot p} \rho_n(p^2) \right.$$

$$\left. + p^\mu e^{i(x-y)\cdot p} \rho_n^*(p^2) \right] =$$

$$= -i \frac{\partial}{\partial y^\mu} \int d^4p \left[\rho_n(p^2) \ e^{-i(x-y)\cdot p} - \rho_n^*(p^2) \ e^{+i(x-y)\cdot p} \right].$$

$$= -i \int dm^2 \frac{\partial}{\partial y^\mu} \left\{ \rho_n(\mu^2) \left[\int \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2) \ e^{-i(x-y)\cdot p} \theta(p_0) \right] \right.$$

$$\left. + \rho_n^*(\mu^2) \left[\int \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2) \ e^{+i(x-y)\cdot p} \theta(p_0) \right] \right\} (2\pi)^3$$

$$= -i(2\pi)^3 \frac{\partial}{\partial y_\mu} \int dm^2 \Big[\rho_n(m^2)\ \Delta_{(+)}(m^2, x-y) - \rho_n^*(m^2)\ \Delta_{(+)}(m^2, y-x) \Big].$$

* <u>Exercise:</u> Prove that for a space-like interval $(x-y)$ $\Delta_+(m^2; x-y) = \Delta_+(m^2, y-x)$.

But, then, the commutator $\langle [J^\mu(y), \phi_n(0)] \rangle$ should vanish. Leading to

$$\rho_n^*(\mu^2) = -\rho_n(\mu^2).$$

<u>Result 1:</u>

$$\langle | [J^\mu(y), \phi_n(0)] \rangle = \frac{\partial}{\partial y_\mu} -i(2\pi)^3 \int dm^2 \rho_n(m^2) \cdot$$

$$\cdot \big[\Delta_{(+)}(m^2, x-y) - \Delta_{(+)}(m^2, y-x) \big]$$

for generic x and y.

Taking a second derivative:

$$\langle 0 | [\partial_\mu J^\mu(y), \phi_n(x)] | 0 \rangle = -i(2\pi)^3 \int dm^2 \rho_n(m^2) \cdot$$

$$\cdot \big[\partial^2 \Delta_{(+)}(m^2, x-y) - \Delta_{(+)}(m^2, y-x) \big] \Rightarrow$$

$$\Rightarrow \quad 0 = \int dm^2 \big[m^2 \rho_n(m^2) \big] \big[\Delta_{(+)}(m^2, x-y) - \Delta_{(+)}(m^2, y-x) \big]$$

$$\rightsquigarrow \quad \boxed{m^2 \rho(m^2) = 0} \qquad (\text{Eq. A})$$

Let's consider $y_0 = x_0 = t$.

$$\langle \,| [J^0(t, \vec{y}), \phi_n(x)] | \rangle =$$

$$= \int dm^2 \rho_n(m^2) \frac{\partial}{\partial y^0} [\Delta_+(m^2, x-y) - \Delta_+(m^2, y-x)]$$

and integrating over $\vec{y}$,

$$\rightsquigarrow \langle \,| [Q, \phi_n(t, \vec{x})] | \rangle =$$

$$= \int dm^2 \rho_n(m^2) \int d^3\vec{y} \int d^4p \, \delta(p^2 - m^2) \sqrt{\vec{p}^2 + m^2} \, [e^{+i\vec{p}\cdot(\vec{x}-\vec{y})} - i e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}]$$

$$= \int dm^2 \rho_n(m^2) \int d^3\vec{y} \, \delta(\vec{y} - \vec{x}) \Rightarrow$$

$$\rightsquigarrow \quad \boxed{\int dm^2 \rho_n(m^2) = T_{nm} \langle \phi_m \rangle} \qquad (\text{Eq. B})$$

For a spontaneously broken symmetry,

$$\langle \phi_n \rangle' = \langle \phi_n \rangle + i\epsilon T_{nm} \langle \phi_m \rangle \neq \langle \phi_n \rangle$$

$$\rightsquigarrow \quad \boxed{T_{nm} \langle \phi_m \rangle \neq 0}$$

→ The "mass spectrum" of the field $\langle \phi_n \rangle$ must have a singularity at $m^2 = 0$, Indeed, for

$$\rho_n(m^2) = \delta(m^2)\left(T_{nm}\langle \phi_m \rangle\right)$$

we have both

$$m^2 \rho_n(m^2) = 0$$

$$\& \quad \int dm^2 \rho_n(m^2) = T_{nm}\langle \phi_m \rangle$$

satisfied !

Let's recall the definition of the density $\rho_n(m^2)$:

$$p^\mu \rho_n(p^2) = \sum_N \langle 0 | J^\mu(0) | N \rangle \langle N | \phi_n(0) | 0 \rangle \delta(p_N - p)$$

It involves a summation of all states. For $\rho_n(p^2) \propto \delta(p^2)$ there must exist a state $|N\rangle$ which has $p_N^2 = 0$!

⇒ The spectrum of the theory has a massless ~~particle~~ state. !

We can get more out of this derivation.

⇒ This massless state is an one-particle state !

* Multi-particle states yield a continuum, not a δ-function.

⇒ This state is invariant under a rotation, i.e it has zero-spin.

Indeed $\phi_\eta(0)|0\rangle$ is invariant under a rotation

$$\phi_\eta(0)|0\rangle \to R(\theta)\phi_\eta(0)|0\rangle =$$
$$= u(0)\phi_\eta(R(\theta)\cdot 0)|0\rangle =$$
$$= u(0)\phi_\eta(0)|0\rangle = \phi_\eta(0)|0\rangle.$$

To preserve $\langle N|\phi_N(0)|0\rangle$ invariant under rotation $\langle N|$ must have zero spin.

⇒ $\langle N|$ must cancel the symmetry transformation of the conserved current $J^a(0)|0\rangle$ ~

~ $|N\rangle$ has the same "quantum numbers" ~~under~~ as $J^a(0)|0\rangle$!