ETH
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The Physics of Electroweak symmetry breaking
Exercise sheet 2

HS 2011

Exercise 2.1 Computing the quantum effective action

In this exercise you are invited to calculate the quantum effective Potential at 1-loop in ϕ^4-theory. The classical action of ϕ^4- theory is

$$S(\phi) = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right].$$

i) Write the scalar field ϕ as $\phi = \phi_{cl} + \eta$, where ϕ_{cl} shall denote the classical field which we assume to be space independent, while η represents quantum fluctuations. Expand the classical action around $\phi = \phi_{cl}$ up to second order, and infer using

$$e^{i\Gamma(\phi_{cl})} = \int_{1PI} [d\eta]\, e^{iS(\phi_{cl}+\eta)}$$

that (let $\Gamma(\phi_{cl}) = \sum_{i\text{loops}} \Gamma^{i\text{loops}}(\phi_{cl})$)

$$\Gamma^{0-\text{loop}}(\phi_{cl}) = -\mathcal{V}_4\left[\frac{m^2}{2}\phi_{cl}^2 + \frac{\lambda}{4!}\phi_{cl}^4\right].$$

Where $\mathcal{V}_4 = \int d^4x = (2\pi)^4\delta^{(4)}(0)$.

ii) Show that the 1-loop correction to $\Gamma(\phi_{cl})$ is given by

$$e^{i\Gamma^{1-\text{loop}}(\phi_{cl})} = \int_{1PI}\left[\prod_x d\eta(x)\right]\exp\left\{\frac{i}{2}\int d^4x\left[\partial_\mu\eta\partial^\mu\eta - M^2\eta^2\right]\right\}$$

where $M^2 = m^2 + \frac{\lambda}{2}\phi_{cl}^2$. Use the identity

$$\int\prod_r d\xi_r \exp\left\{-\frac{1}{2}\sum_{rs}K_{rs}\xi_r\xi_s\right\} = \left[\det\left(\frac{K}{2\pi}\right)\right]^{-1/2}$$

to show that

$$i\Gamma^{1-\text{loop}} = \ln\det\left(-\frac{iK}{2\pi}\right)^{-1/2} = -\frac{1}{2}\mathrm{Tr}\ln\frac{-iK_{xy}}{2\pi}$$

The matrix K_{xy} can be diagonalised by fourier transformation, show that it's Fourier Transform is

$$K_{pq} = \left(p^2 - M^2\right)\delta^{(4)}(p-q).$$

and hence that (promoting the dimension from 4 to D, and remembering about the $i\varepsilon$ prescription)

$$\Gamma^{1-\text{loop}} = -\frac{\mathcal{V}_4}{2(2\pi)^4}\int d^Dp\ln\left(\frac{-i(p^2-M^2-i\varepsilon)}{2\pi}\right) = -\frac{\mathcal{V}_4(M^2)^{d/2}\Gamma\left(-\frac{d}{2}\right)}{2(4\pi)^{d/2}}.$$

Let $\lambda \to \lambda + \delta\lambda$ and $m \to m + \delta m$ and find a minimal choice for $\delta\lambda$ and δm to cure the divergences in $\Gamma(\phi_{cl})$. For further reading see for example Peskin & Schroeder Pg.373, as well as Weinberg II Pg. 70.