Exercise 12.1 Gaussian Fluctuations in the Ginzburg-Landau Model

Consider the Ginzburg-Landau model of the d-dimensional Ising model in presence of a magnetic field $H(\vec{r})$, introduced in chapter 5.3 of the lecture notes. Here, we only consider temperatures above the critical temperature T_c. In order to make the model exactly tractable, we assume that quartic fluctuations are negligible and ignore them. Therefore, the free energy functional for a given magnetization m and temperature T in d dimensions is given by

$$F(T,m,H) \; = \; \frac{1}{2}\int d^d r \left\{ A\, m(\vec{r})^2 - H(\vec{r})\, m(\vec{r}) + \kappa\big[\vec{\nabla} m(\vec{r})\big]^2 \right\}, \tag{1}$$

where $A = a\tau$, with $\tau = (T - T_c)/T_c$. For the calculations we assume our system to be a cube of side length L with periodic boundary conditions on m.

a) Use the Fourier transform of the magnetization field,

$$m(\vec{r}) \; = \; \frac{1}{\sqrt{L^d}} \sum_{\vec{q}} m_{\vec{q}}\, e^{i\vec{q}\cdot\vec{r}} \,, \tag{2}$$

and compute the energy functional $F(T, m)$ in the transformed coordinates $\{m_{\vec{q}}\}$. Which values of $\vec{q}$ are allowed in the sum and which values of $\vec{q}$ are independent? Note that $m(\vec{r})$ is real and interpret its implication on the $m_{\vec{q}}$.

b) Compute the canonical partition function,

$$Z(T) \; = \; \int \mathcal{D}m \; e^{-F(T,m)/k_B T} \,, \tag{3}$$

and the free energy $F(T) = -k_B T \log Z(T)$ by using Gaussian integration. Argue that the finite number of degrees of freedom (finite lattice spacing) of our Ising model introduces a momentum cutoff Λ, which is crucial to regulate the otherwise ill defined integrals (cf. Debye wave vector for phonons).

Hint: Rewrite the functional measure $\mathcal{D}m$ according to

$$\mathcal{D}m = \prod_{\vec{q}} dm_{\vec{q}}\, dm_{-\vec{q}} \,. \tag{4}$$

Why do we use $dm_{\vec{q}}\, dm_{-\vec{q}}$?

c) Compute the internal energy and the specific heat c_V in the thermodynamic limit $L \to \infty$ for vanishing external field ($H(\vec{r}) \equiv 0$). Study its behavior near the critical temperature where $\tau = 0$. Compare the critical exponent of c_V with the mean field result of section 5.3.2 for different dimensions d.

d) Derive an expression for the magnetic susceptibility, defined as the negative second derivative of the free energy with respect to the external field H in the limit of vanishing field, i.e.

$$\chi(T) = -\left.\frac{\partial^2 F(T)}{\partial H^2}\right|_{H=0} . \tag{5}$$

What is the critical exponent of χ? Compare the result with the mean field result of section 5.2.5 of the lecture notes, Eq. (5.58).

Office Hours: Monday, 13th December, 3–5 pm (HIT K 31.3)