# Statistical Physics
# Exercise Sheet 5



## Exercise 5.1 Drude Conductivity

The goal of this exercise is to extend the calculation of the conductivity of an electron gas in the relaxation time approximation (as discussed in the lecture) to the case of a time-dependent electric field, $\vec{E}(t) = \vec{E}_0\, e^{-i\omega t}$.

a) Split the distribution function into an equilibrium distribution $f_0(\vec{p})$ and a small perturbation $g(\vec{p}, t)$, i.e.

$$f(\vec{p}, t) = f_0(\vec{p}) + g(\vec{p}, t)\,, \tag{1}$$

and find the Boltzmann equation for $g(\vec{p}, t)$. Justify that the solution is of the form $g(\vec{p}, t) = g_\omega(\vec{p})\, e^{-i\omega t}$ and find $g_\omega(\vec{p})$. Furthermore, find expressions for the current response $\vec{j}_\omega$ and the conductivity $\sigma_\omega$ (defined by $\vec{j}_\omega = \sigma_\omega \vec{E}_0$).

b) Assume a Maxwell-Boltzmann distribution for $f_0$ and calculate the time-dependent current $\vec{j}(t)$ for an external field given by $\vec{E}(t) = \vec{E}_0 \cos\omega t$. Show that in the limiting case $\omega \to 0$ the Drude conductivity is recovered.

c) From the result of b), calculate

$$p = \langle \vec{j} \cdot \vec{E} \rangle_t = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \mathrm{d}t\ \vec{j}(t) \cdot \vec{E}(t)\,. \tag{2}$$

This term describes Ohmic heating (it is the local version of $P = UI$), which is accompanied by an increase in entropy. However, it is easy to verify that $H(t) = H(t + \frac{2\pi}{\omega})$. What assumptions are made in a) and b) and where does the entropy production take place if these assumptions are justified?

d) The derivation of the Drude conductivity in a) and b) used some drastic approximations. Unfortunately, for most interesting systems, this is the usual state of affairs rather than an exception. However, for response functions, of which the conductivity is an example, the Kramers-Kronig relations

$$\mathrm{Re}\{\sigma_\omega\} = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \mathrm{d}\nu\, \frac{\mathrm{Im}\{\sigma_\nu\}}{\nu - \omega}\,, \tag{3}$$

$$\mathrm{Im}\{\sigma_\omega\} = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \mathrm{d}\nu\, \frac{\mathrm{Re}\{\sigma_\nu\}}{\nu - \omega} \tag{4}$$

can be used to check the plausibility of the approximation. As these relations are a consequence of causality, any reasonable approximation of the conductivity should satisfy them. Verify that the result obtained in b) fulfills the Kramers-Kronig relations.

## Exercise 5.2 Microcanonical Ensemble: Harmonic Potential with Uniform External Force

We consider an ideal gas of $N$ non-interacting particles in a harmonic trap, described by the following Hamilton function:

$$\mathcal{H}(p,q) = \sum_{i=1}^{N} \left\{ \frac{\vec{p}_i^{\,2}}{2m} + \frac{D\vec{q}_i^{\,2}}{2} \right\} . \tag{5}$$

a) Using the microcanonical ensemble, calculate the entropy of this system and the caloric equation of state, i.e. $S(U,...)$ and $U(T,...)$, respectively.

   *Hint:* The volume of an $n$-dimensional sphere is given by $C_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$, where $\Gamma(z) \approx (2\pi)^{1/2} e^{-z} z^{z-1/2}$ for $z \gg 1$ in the Stirling approximation.

b) Now assume that the particles have charge $e$ (miraculously still not interacting with each other) and apply an external electric field $\vec{E}$ leading to the following correction to the Hamilton function:

$$\Delta\mathcal{H}(p,q) = -e \sum_{i=1}^{N} \vec{E} \cdot \vec{q}_i \,. \tag{6}$$

   Determine the entropy and the caloric equation of state of the gas including $\Delta\mathcal{H}(p,q)$. What is the dielectric polarization of system, $\vec{P}$, as a function of the electric field?

   *Hint:* Note that $dU = \cdots + \vec{P} d\vec{E}$.

**Office Hours:** Monday, 25.10, 9 – 11 at HIT K 11.3.