

Exercise 4.1 Local equilibrium state of a gas in a periodic potential

We consider a gas of N particles trapped in a box, $\vec{r} \in V = [0, L]^3$, in the presence of a conservative force $\vec{F}(\vec{r}) = -\nabla V(\vec{r})$ originating from a periodic potential in the x -direction,

$$V(\vec{r}) = V_0 \cos(2\pi kx/L), \quad k \in \mathbb{N}. \quad (1)$$

For the distribution function in equilibrium, we make the following ansatz; we generalize the free-particle (Maxwell-Boltzmann) distribution function by taking into account a local density $n(\vec{r})$, i.e.,

$$f_0(\vec{r}, \vec{p}) = \frac{n(\vec{r})}{(2\pi m k_B T)^{3/2}} e^{-\beta p^2/2m}, \quad \beta = \frac{1}{k_B T}. \quad (2)$$

- Find the local density $n(\vec{r})$. Discuss the limits $\beta V_0 \ll 1$ and $\beta V_0 \gg 1$.
- Determine the internal energy U and the specific heat C_V . Discuss these expressions in the limits $\beta V_0 \ll 1$ and $\beta V_0 \gg 1$.
- Calculate the entropy $S = S(T, V, N)$.

Hints: The integral representation and the series expansion of the modified Bessel functions of the first kind for $n \in \mathbb{Z}$ are

$$I_n(z) = \frac{1}{\pi} \int_0^\pi d\theta e^{z \cos \theta} \cos(n\theta) = \left(\frac{z}{2}\right)^n \sum_{k \geq 0} \frac{(z^2/4)^k}{k!(n+k)!}.$$

The asymptotic behavior for $z \rightarrow \infty$ is

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left[1 - \frac{4n^2 - 1}{8z} + \dots \right].$$

Furthermore, the relation $I'_0(z) = I_1(z)$ holds.

Exercise 4.2 Distribution function for relativistic massive particles

Consider a system of relativistic massive particles with vanishing drift velocity $\vec{v}_{\text{drift}} = 0$. The energy-momentum relation of the particles writes

$$E(\vec{p}) = \sqrt{p^2 c^2 + m^2 c^4}, \quad (3)$$

where m is the particle mass and c the speed of light.

- Find the equilibrium distribution function $f_0(\vec{p})$ and show that in the limit $k_B T \ll mc^2$, the classical Maxwell-Boltzmann distribution function is recovered. Calculate the internal energy U and determine the first relativistic corrections to this expression as well as to the specific heat C_V .

- b) Calculate the pressure P for this system and convince yourself that in the limit $k_B T/mc^2 \ll 1$, it is consistent with the pressure of a dilute ideal gas as derived in the lecture notes.
- c) Take into account a finite drift velocity \vec{v}_{drift} . How does the distribution function change? Does the temperature depend on the drift velocity? Calculate the average momentum $\langle \vec{p} \rangle$ for this system.

Hints: The integral representation of the modified Bessel functions of the second kind is

$$K_n(z) = \int_0^\infty dy e^{-z \cosh y} \cosh(ny),$$

Its asymptotic behavior for $z \rightarrow \infty$ is

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{4n^2 - 1}{8z} + \frac{(4n^2 - 1)(4n^2 - 9)}{2!(8z)^2} + \frac{(4n^2 - 1)(4n^2 - 9)(4n^2 - 25)}{3!(8z)^3} + \dots \right].$$

Office Hours: Monday, October 18, 9-11 am (HIT K 11.3)