ETH
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Statistical Physics
Exercise Sheet 3

HS 10
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Exercise 3.1   Relaxation of Polarized Spins Coupled to a Bath

Consider a system S of $N^{(S)}$ units with two states (spins $\uparrow^{(S)}$ and $\downarrow^{(S)}$) that are coupled to a reservoir R consisting of other two-state units (labeled as $\uparrow^{(R)}$ and $\downarrow^{(R)}$) and is infinitely larger than S. This is a simplified model for nuclear spins in a metal, whereby the nuclear spins represent our system and the electrons (electron spins) the reservoir.

We assume that the microstates in S and R are non-degenerate with an energy difference of $\Delta\epsilon$,

$$E_\uparrow^{(S)} - E_\downarrow^{(S)} = E_\uparrow^{(R)} - E_\downarrow^{(R)} = \Delta\epsilon , \tag{1}$$

where $E_\uparrow^{(S)}$ and $E_\downarrow^{(S)}$ ($E_\uparrow^{(R)}$ and $E_\downarrow^{(R)}$) are the energies for the two states in S (R).

The reservoir R is in its thermodynamic equilibrium such that the occupation probabilities $w_\uparrow^{(R)}$ and $w_\downarrow^{(R)}$ of the two states satisfy the relation,

$$\frac{w_\uparrow^{(R)}}{w_\downarrow^{(R)}} = \exp\left\{-\frac{\Delta\epsilon}{k_B T^{(R)}}\right\} , \tag{2}$$

where $T^{(R)}$ defines the temperature of R. At the beginning, $t = 0$, S shall not be in equilibrium with R.

There is a coupling between the units of S and R in the following way. A spin of S can only be reversed, if simultaneously a spin of R is flipped in the opposite way, i.e: $\uparrow^{(S)}\rightarrow\downarrow^{(S)}$ together with $\downarrow^{(R)}\rightarrow\uparrow^{(R)}$ and $\downarrow^{(S)}\rightarrow\uparrow^{(S)}$ together with $\uparrow^{(R)}\rightarrow\downarrow^{(R)}$. This corresponds to spin conservation as well as energy conservation.

Both types of processes occur with the same probability Γ (time reversal invariance):

$$|\ldots,\uparrow,\ldots\rangle^{(S)} \otimes |\ldots,\downarrow,\ldots\rangle^{(R)} \xrightarrow{\Gamma} |\ldots,\downarrow,\ldots\rangle^{(S)} \otimes |\ldots,\uparrow,\ldots\rangle^{(R)} \tag{3}$$

$$|\ldots,\downarrow,\ldots\rangle^{(S)} \otimes |\ldots,\uparrow,\ldots\rangle^{(R)} \xrightarrow{\Gamma} |\ldots,\uparrow,\ldots\rangle^{(S)} \otimes |\ldots,\downarrow,\ldots\rangle^{(R)} \tag{4}$$

From this consideration the effective transition rates ($\Gamma_{\uparrow\downarrow}$, $\Gamma_{\downarrow\uparrow}$) for the spins in S depend on the probabilities of finding states $\uparrow^{(R)}$ and $\downarrow^{(R)}$ in R, such that

$$\Gamma_{\uparrow\downarrow} = \Gamma w_\downarrow^{(R)} , \qquad \Gamma_{\downarrow\uparrow} = \Gamma w_\uparrow^{(R)} . \tag{5}$$

a) Formulate the master equation for the system S where $N_\uparrow^{(S)}$ and $N_\downarrow^{(S)}$ denote the occupation numbers of the two spin states ($N_\uparrow^{(S)} + N_\downarrow^{(S)} = N^{(S)}$). What does the condition of detailed balance imply?

b) Compute the time dependent polarization of the localized spin system, defined by $P(t)^{(S)} = (N_\uparrow^{(S)}(t) - N_\downarrow^{(S)}(t))/N^{(S)}$, for the following starting conditions at $t = 0$:

   (i)   unpolarized case ($N_\uparrow^{(S)} = N_\downarrow^{(S)}$),

   (ii)   almost complete polarization in $\downarrow$-direction ($N_\downarrow^{(S)} = N^{(S)}(1-\delta)$, $N_\uparrow^{(S)} = N^{(S)}\delta$, $\delta \ll 1$),

(iii) almost complete polarization in $\uparrow$-direction ($N_\uparrow^{(S)} = N^{(S)}(1-\delta)$, $N_\downarrow^{(S)} = N^{(S)}\delta$, $\delta \ll 1$)

Determine the relaxation time for these three cases. Define a temperature of the spin system $T^{(S)}$ for the three initial conditions by using a relation analogous to the definition of the temperature of the reservoir, Eq. (2).

c) Calculate the H-function of S and its derivative with respect to time for the three cases in b). Moreover, calculate the energy exchange between S and R as a function of t (take the temperature of the reservoir $T^{(R)}$ as a parameter) for the same three cases. How does the temperature $T^{(S)}$ evolve as a function of time? Calculate the specific heat of S as a function of temperature $T^{(S)}$.

Exercise 3.2   The Linear Boltzmann Equation

We want to study the Boltzmann equation when only one-particle scattering is considered, contrary to two-particle scattering as presented in the lecture. Let us consider a system of particles imbedded in a static background of impurities such that one particle with the momentum $\vec{p}$ has a certain probability to be scattered to a new state with the momentum $\vec{p}'$. We assume that the scattering processes are instantaneous and elastic and that the scatterers are isotropic.

a) Following the approach of the lecture notes, derive the Boltzmann equation of the system.

b) List the quantities which are conserved in the one-particle collision and verify the relation

$$\int d^3p\, \varphi(\vec{p}) \left(\frac{\partial f(\vec{r}, \vec{p}, t)}{\partial t}\right)_{\text{coll}} = 0 \,, \tag{6}$$

where $\varphi(\vec{p})$ denotes a conserved quantity (we use the notations of the lecture notes).

c) Prove the "H-theorem" discussing the assumptions you make. Find the equilibrium state and give the physical interpretation of a relaxation process towards equilibrium (what does it change in the system?).

Office Hours: Monday, October 11, 10-12 am (HIT K 12.2)