Exercise 1) Universality of Fibonacci Anyons

Define $R = \begin{pmatrix} \exp(4\pi i/5) & 0 \\ 0 & -\exp(2\pi i/5) \end{pmatrix}$ and $B = F^{-1}RF$, where $F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$ and $\tau = \frac{\sqrt{5}-1}{2}$. First note that $R, F \in U(2)$ but $R, F \notin SU(2)$, since $\det(R) = \det(B) = \exp(\frac{\pi i}{5})$.

But R and B are both of order 10 in $U(2)$ and because the overall phase of a quantum state is unobservable, we will just show that the subgroup generated by $\{R, B\}$ closes on the group containing all elements of $U(2)$, but with a determinant equal to a 10^{th} root of unity (instead of a determinant equal to 1).

The homomorphism from $U(2)$ to $SO(3)$ (see page 6 in the script) maps finite subgroups of $U(2)$ to finite subgroups of $SO(3)$. So if we could show that the image of $\{R, B\}$ under this homomorphism generates an infinite subgroup of $SO(3)$, then $\{R, B\}$ would generate an infinite subgroup of $U(2)$.

But R and B are elements of order 10 in $U(2)$ about two distinct axis and therefore the subgroup generated by $\{R, B\}$ closes on the group containing all elements of $U(2)$, with a determinant equal to a 10^{th} root of unity.

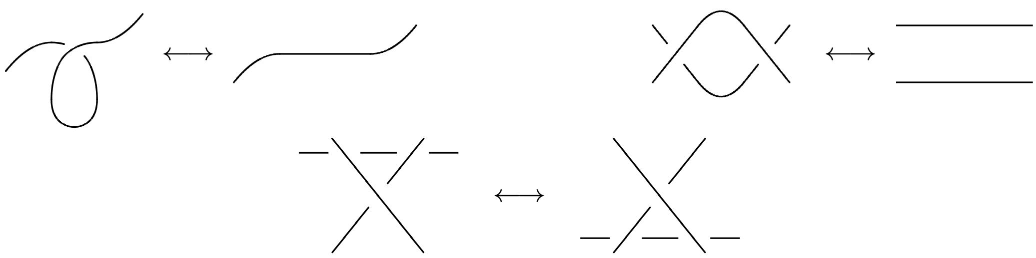
Hence it remains to show that the image of $\{R, B\}$ under the homomorphism from $U(2)$ to $SO(3)$ generates an infinite subgroup of $SO(3)$. For this we use the classification of finite subgroups of $SO(3)$: (1) C_k the cyclic group, (2) D_k the dihedral group, (3) T the tetrahedral group, (4) O the octahedral group and (5) I the icosahedral group.

Since the image of R and B do not commute in $SO(3)$ they can not belong to (1) or (2) and since the image of R and B are of order 10 in $SO(3)$, they can not belong to (3)-(5) either (by inspection all the subgroups (3)-(5) do not have elements of order > 5).

This concludes the proof.

Exercise 2) Knot Theory

(i) Two (oriented) links (or as a special case knots) are equivalent if and only if the are related by Reidemeister moves:



By this criterion one can see by inspection that the unknot and the trefoil knot are not equivalent.

(ii) By Theorem 5 on page 48 in the script every oriented link can be written as a closure of a braid (Alexander). And by Theorem 6 on page 48 in the script two oriented links are equivalent if and only if the corresponding braids are equivalent (Markov). This transforms the problem of testing equivalence of links to testing equivalence of braids.

A general way to test equivalence is to associate an invariant to each (oriented) link, that is an algebraic object that stays invariant under Reidemeister moves. If two (oriented) links have different objects associated to them, then they cannot be equivalent. Examples are the Jones polynomial or the Alexander polynomial.