Exercise 1) Permutation Group S_n

We can reach any permuation $\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$ with the following algorithm. We start with the permutation

$$\pi_0 \equiv \mathrm{id} = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} .$$

Then we take k_1 with $\pi(k_1) = 1$. By transposing pairs of neighbors $(k_1 - 1)$ times we can get

$$\pi_1 = \begin{pmatrix} 1 & 2 & \dots & k_1 - 1 & k_1 & k_1 + 1 & \dots & n \\ 2 & 3 & \dots & k_1 & 1 & k_1 + 1 & \dots & n \end{pmatrix} .$$

Since $k_1 \leq n$ we need at most $O(n)$ transpositions of pairs of neighbors to achieve this. As a next step we do this for k_2 with $\pi(k_2) = 2$.[1]

Reiterating this until k_n with $\pi(k_n) = n$, we get the desired permutation π. Since there are n elements, a product of at most $O(n^2)$ transpositions of pairs of neighbors is sufficient.

Exercise 2) Two-Qubit Gate

Using the algorithm given on page 34 in the script we can get a decomposition of U into two-level unitaries. We get $U = U_1 U_2 U_3 U_4$ with

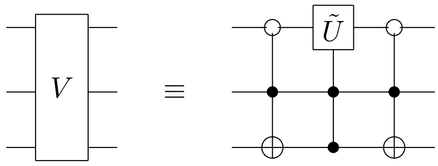
$$U_1 = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \ U_2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix} ,$$

$$U_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1+i)\frac{\sqrt{3}}{4} & \frac{1}{4}(3-i) & 0 \\ 0 & \frac{1}{4}(3-i) & -(1+i)\frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \ U_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & -\frac{i}{\sqrt{3}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{i}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix} .$$

Exercise 3) Two-Level Unitary

V acts non trivially on the subspace spanned by $|2\rangle := |010\rangle$ and $|7\rangle := |111\rangle$. We need to transform V such that the non-trivially transformed subspace is spanned by two vectors that differ in one digit only.

For example, we can flip the states $|2\rangle = |010\rangle$ and $|3\rangle := |011\rangle$. This can be achieved by flipping the third qubit iff the first qubit is $|0\rangle$ and the second is $|1\rangle$. Hence we can implement V as follows:



[1] Note that if $k_2 > k_1$, the element $\pi(k_1) = 1$ is shifted one to the left by the steps induced by k_2. But this can be adjusted again by just one additional transposition of pairs of neighbors.