**Exercise 1) Clebsch-Gordan Coefficients**

**a)** By the Schur transform we have

$$V_1^{\otimes n} = \bigoplus_k V_k \otimes \mathbb{C}^{m_k^n} \ ,$$

where $m_k^n = \binom{n}{\frac{n-k}{2}} \cdot \frac{2k+2}{n+k+2}$. For $n = k$ we get $m_k^k = 1$ and hence $V_k$ is a sub-representation of $V_1^{\otimes k}$ that appears with multiplicity one.

Because the weight of the vectors is additive we need to have $l$ ones and therefore $k-l$ zeros for the vector $|k,l\rangle$. Furthermore, since the action of $SU(2)$ and $S_k$ commute, we have

$$|k,l\rangle \propto |\underbrace{11\ldots1}_{l}\underbrace{00\ldots0}_{k-l}\rangle + \text{perm} \ .$$

Since $|k,l\rangle$ has to be normalized to one, we can conclude

$$|k,l\rangle = \frac{1}{\sqrt{\binom{k}{l}}}\left(|\underbrace{11\ldots1}_{l}\underbrace{00\ldots0}_{k-l}\rangle + \text{perm}\right) \ .$$

**b)** The weight of the state on the LHS is given by $2\cdot l-(k+1) = 2l-k-1$ and the weight of the states on the RHS is given by $2\cdot l_1 + 2\cdot 1 - k - 1 = 2l_1 - k + 1$ and $2\cdot l_2 + 2\cdot 0 - k - 1 = 2l_2 - k - 1$ resp. Since the states on the RHS have the same weight as the state on the LHS we get $l_1 = l-1$ and $l_2 = l$.

Using the formula found in a) we can write the state on the RHS as

$$|k+1,l\rangle = \frac{1}{\sqrt{\binom{k+1}{l}}}\left(|\underbrace{11\ldots1}_{l}\underbrace{00\ldots0}_{k+1-l}\rangle + \text{perm}\right) \ ,$$

and the first state on the LHS as

$$|k,l-1\rangle \otimes |1,1\rangle = \frac{1}{\sqrt{\binom{k}{l-1}}}\left(|\underbrace{11\ldots1}_{l-1}\underbrace{00\ldots0}_{k-l+1}\rangle + \text{perm}\right) \otimes |1\rangle \ .$$

Hence we can get

$$\begin{aligned} c_1 &= \langle k+1,l|(|k,l-1\rangle \otimes |1,1\rangle) \\ &= \frac{1}{\sqrt{\binom{k+1}{l}\cdot\binom{k}{l-1}}}(\langle\underbrace{11\ldots1}_{l}\underbrace{00\ldots0}_{k+1-l}| + \text{perm})((|\underbrace{11\ldots1}_{l-1}\underbrace{00\ldots0}_{k-l+1}\rangle + \text{perm})\otimes|1\rangle) \\ &= \frac{\binom{k}{l-1}}{\sqrt{\binom{k+1}{l}\cdot\binom{k}{l-1}}} = \sqrt{\frac{l}{k+1}} \ . \end{aligned}$$

By similar arguments (or just normalization) we can also get $c_2 = \sqrt{\frac{k-l+1}{k+1}}$.

So we can conclude that

$$|k+1,l\rangle = \sqrt{\frac{l}{k+1}}|k,l-1\rangle \otimes |1,1\rangle + \sqrt{\frac{k-l+1}{k+1}}|k,l\rangle \otimes |1,0\rangle \ .$$

**c)** By the orthogonality of $|k-1, l-1\rangle$ and $|k+1, l\rangle$ and using the same phase convention as in the script we can get

$$|k-1, l-1\rangle = -\sqrt{\frac{k+1-l}{k+1}}|k, l-1\rangle \otimes |1, 1\rangle + \sqrt{\frac{l}{k+1}}|k, l\rangle \otimes |1, 0\rangle \ .$$

## Exercise 2) Representations of the Symmetric Group

**a)** The action of $S_2$ on $(\mathbb{C}^2)^{\otimes 2}$ can be written as (in the standard basis):

$$\mathrm{id}_{12} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \ \pi_{12} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

By the Schur transform we have

$$V_1 \otimes V_1 \cong (V_0 \otimes \mathbb{C}^1) \oplus (V_2 \otimes \mathbb{C}^1) \ .$$

So if we decompose the representation of $S_2$ on $(\mathbb{C}^2)^{\otimes 2}$ into a direct sum of irreducible representation, we know by the Schur-Weyl duality that two one-dimensional representations appear, one with multiplicity one and one with multiplicity three.

We can diagonalize the matrix that corresponds to $\pi_{12}$ and get the eigenvalues $(1, 1, 1, -1)$ with corresponding eigenvectors $|a\rangle := |00\rangle$, $|b\rangle := |11\rangle$, $|c\rangle := \frac{1}{\sqrt{2}}|01 + 10\rangle$ and $|d\rangle := \frac{1}{\sqrt{2}}|01 - 10\rangle$.

Hence on the subspace spanned by $|a\rangle$ the action of $S_2$ is trivial and the same happens on the subspace spanned by $|b\rangle$ and the subspace spanned by $|c\rangle$. On the subspace spanned by $|d\rangle$ the action of $S_2$ is given by

$$\mathrm{id}_{12} \mapsto 1, \ \pi_{12} \mapsto -1 \ ,$$

the alternating representation.

So the one-dimensional trivial representation appears with multiplicity three and the one-dimensional alternating representation appears with multiplicity one.

**b)** It is enough to specify the action of $S_3$ on $(\mathbb{C}^2)^{\otimes 3}$ on a generating set of $S_3$. We choose the set $\{\pi_{12} \otimes \mathrm{id}_3, \mathrm{id}_1 \otimes \pi_{23}\}$. The action of $S_3$ on $(\mathbb{C}^2)^{\otimes 3}$ can now be written as (in the standard basis):

$$\pi_{12} \otimes \mathrm{id}_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathbb{1}_2 \ , \mathrm{id}_1 \otimes \pi_{23} \mapsto \mathbb{1}_2 \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

By the Schur transform we have

$$V_1^{\otimes 3} \cong (V_1 \otimes \mathbb{C}^2) \oplus (V_3 \otimes \mathbb{C}^1) \ .$$

So if we decompose the representation of $S_3$ on $(\mathbb{C}^2)^{\otimes 3}$ into a direct sum of irreducible representation, we know by the Schur-Weyl duality that a one-dimensional representation with multiplicity four and a two-dimensional representation with multiplicity two appear.

The one-dimensional representation that appears is given by the trivial representation and the corresponding one-dimensional subspaces are given by: $|a\rangle := |000\rangle$, $|b\rangle := |111\rangle$, $|c\rangle := \frac{1}{\sqrt{3}}|001+010+100\rangle$ and $|d\rangle := \frac{1}{\sqrt{3}}|011+101+110\rangle$.

To specify the first two-dimensional representation, we use

$$|e\rangle := |1,0,-+\rangle = \frac{1}{\sqrt{2}}|01-10\rangle \otimes |0\rangle \equiv \frac{1}{\sqrt{2}}|010-100\rangle \ ,$$

as on pages 20-21 in the script. To get a second vector we note that the action of $SU(2)$ commutes with the action of $S_3$ and hence for any $\pi \in S_3$: $\pi|1,0,-+\rangle = \sum_p c_p|1,0,p\rangle$. The element $\pi_{12} \otimes \mathrm{id}_2$ just gives a minus one, but for $\mathrm{id}_1 \otimes \pi_{23}$ we get

$$|f'\rangle := (\mathrm{id}_1 \otimes \pi_{23})\frac{1}{\sqrt{2}}|010-100\rangle = \frac{1}{\sqrt{2}}|001-100\rangle \ .$$

This new vector $|f'\rangle$ is not orthogonal to $|e\rangle$, but using Gram-Schmidt orthonormalization we can get such a vector:

$$|f\rangle := \sqrt{\frac{2}{3}}|001\rangle - \frac{1}{\sqrt{6}}|010\rangle - \frac{1}{\sqrt{6}}|100\rangle \ .$$

The space spanned by $\{|e\rangle, |f\rangle\}$ now gives the space of the first two dimensional representation. The action in the basis $\{|e\rangle, |f\rangle\}$ can be calculated to

$$\mathrm{id}_1 \otimes \pi_{23} \mapsto \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \ \pi_{12} \otimes \mathrm{id}_3 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

To specify the second two-dimensional representation, we use

$$|g\rangle := |1,1,-+\rangle = \frac{1}{\sqrt{2}}|01-10\rangle \otimes |1\rangle \equiv \frac{1}{\sqrt{2}}|011-101\rangle \ .$$

Similarly as before we can get

$$|h'\rangle := (\mathrm{id}_1 \otimes \pi_{23})\frac{1}{\sqrt{2}}|011-101\rangle = \frac{1}{\sqrt{2}}|011-110\rangle \ .$$

as well as

$$|h\rangle := \frac{1}{\sqrt{6}}|011\rangle + \frac{1}{\sqrt{6}}|101\rangle - \sqrt{\frac{2}{3}}|110\rangle \ .$$

The space spanned by $\{|g\rangle, |h\rangle\}$ now gives the space of the second two dimensional representation. The action in the basis $\{|g\rangle, |h\rangle\}$ can be calculated to

$$\mathrm{id}_1 \otimes \pi_{23} \mapsto \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \ \pi_{12} \otimes \mathrm{id}_3 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

the same matrices as before (as it should be since the same two-dimensional representation appears two times).