Exercise 1) S_4 in Schur-Weyl Duality

Write down the matrices of the representation of S_4 on $(\mathbb{C}^2)^{\otimes 4}$. Decompose this representation into a direct sum of irreducible representation and show explicitly how S_4 acts on the direct summands of this decomposition.

Exercise 2) The Sign Representation of S_n

Apart from the trivial (and in particular one-dimensional) representation of S_n given by

$$S_n \ni \pi \mapsto 1 \in \mathbf{C}$$

there is a second one-dimensional representation of S_n given by

$$S_n \ni \pi \mapsto \mathrm{sign}(\pi) \in \mathbf{C}$$

where $\mathrm{sign}(\pi)$ is the sign of the permutation. When $n = 2$, we have found this representation once in

$$(\mathbf{C}^2)^{\otimes 2}$$

as the singlet. In general it appears once in

$$(\mathbf{C}^n)^{\otimes n}.$$

(Just as in $(\mathbf{C}^2)^{\otimes n}$, the permutation group acts by permuting the tensor factors). Write down the vector that spans this irreducible representation.