Exercise 1) Rotations on the Bloch sphere

Let $\vec{e}$ be a unit vector, $\alpha \in [0, 4\pi)$ and define $U(\vec{e}, \alpha) = \exp(-i\frac{\alpha}{2}\vec{e}\cdot\vec{\sigma})$ as in the lecture. Show that when $U(\vec{e}, \alpha)$ is applied to a density matrix (by conjugation), this results in a rotation $R(\vec{e}, \alpha)$ of the Bloch vector around the axis $\vec{e}$ with an angle α in the Bloch sphere. That is, show the formula

$$U(\vec{e}, \alpha)(\vec{v}\cdot\vec{\sigma})U(\vec{e}, \alpha)^\dagger = (R(\vec{e}, \alpha)\vec{v})\cdot\vec{\sigma} \ .$$

Exercise 2) Quantum Teleportation

Recall the protocol for teleportation from the lecture.

1. Why does teleportation not allow to transmit quantum states faster than light?
2. Show that teleportation also works for mixed state inputs.
3. Imagine that the two parties do not have access two an ebit. Is it possible to transmit quantum states only having a classical communication channel?
4. In quantum information theory, there is the so called "no-cloning theorem". It states that it is in general impossible to make a copy of an unknown quantum state. Why does teleportation not contradict this theorem?

Exercise 3) Entanglement Swapping

Assume that Alice and Charlie share an ebit and that Bob and Charlie do. Come up with a protocol that – at the expense of the other two ebits – allows Alice and Bob to share an ebit. The protocol should only involve local operations (measurements, unitaries at Alice, Bob or Charlie side) and classical communication.

What could this protocol, known as *entanglement swapping* be useful for?

Exercise 4) Representations of SU(2)

Let the notation be as in the script and $0 \leq l \leq k$. Verify that

$$v_k(\sigma_-)|k, l\rangle = \sqrt{l(k-l+1)}|k, l-1\rangle$$
$$v_k(\sigma_+)|k, l\rangle = \sqrt{(k-l)(l+1)}|k, l+1\rangle$$
$$v_k(\sigma_z)|k, l\rangle = (2l-k)|k, l\rangle$$

defines a representation of of the Lie algebra $\mathrm{su}(2)_{\mathbb{C}}$.

Show also that

$$\sum_{i\in\{x,y,z\}} v_k(\sigma_i)v_k(\sigma_i) = k(k+2)\mathbb{1}.$$

Apply the lowering operator and use induction in order to prove that

$$|k+1, l\rangle = \sqrt{\frac{l}{k+1}}|k, l-1\rangle|1, 1\rangle + \sqrt{\frac{k+1-l}{k+1}}|k, l\rangle|1, 0\rangle.$$