VI. SUPERSPACE AND SUPERFIELDS

The direct-construction procedure to build supermultiplets is always possible, however it becomes usually very cumbersome, especially when we must build susy Lagrangians. What we did to build a Lagrangian was just to guess its form and then explicitly check susy invariance.

Fortunately for N=1 susy a very powerful method has been found to construct susy multiplets and susy Lagrangian. This method consists in encoding all the components of a susy multiplet in a single object, the "superfield", which is a field in an extended "superspace" that includes the usual space-time coordinates plus some extra fermionic coordinates. Superfields can be easily combined to form new superfields and this property can be used to extract susy invariant quantities from which we can build a susy Lagrangian.

In the following we will first introduce the concept of superspace and then we will see how to construct superfields defined on it.

· The N = 1 rigid superspace

As the four-momentum operators P_μ are defined as the generators of translations of the ordinary spacetime coordinates x^μ, the supersymmetry generators Q_α and $\bar{Q}_{\dot\alpha}$ may be regarded as the generators of "translations" of some fermionic superspace coordinates θ_α and $\bar\theta_{\dot\alpha}$, which anticommute with each other and with fermionic fields and commute with the x^μ and all bosonic fields. The susy generators have non-vanishing anticommutators, so we can not take them as simply proportional to the supercoordinate translation operators.

To find the correct representation of the susy generators we start from the observation that the susy algebra may be viewed as a Lie algebra with anticommuting parameters. This motivates us to define a corresponding group element

$$G(x,\theta,\bar\theta) = e^{i(-x^\mu P_\mu + \theta Q + \bar\theta\bar Q)}.$$

To multiply group elements we can use the Hausdorff's formula in the form

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]},$$

which comes from the fact that higher commutators vanish for the P_μ, Q_α and $\bar Q_{\dot\alpha}$ generators. In this way we get

$$G(0,\varepsilon,\bar\varepsilon)\,G(x^\mu,\theta,\bar\theta) = G(x^\mu + i\theta\sigma^\mu\bar\varepsilon - i\varepsilon\sigma^\mu\bar\theta,\ \theta+\varepsilon,\ \bar\theta+\bar\varepsilon).$$

This means that a group element $G(0,\varepsilon,\bar{\varepsilon})$ induces the following motion on the parameter space

$$(x^\mu,\theta,\bar{\theta}) \xrightarrow{G(0,\varepsilon,\bar{\varepsilon})} (x^\mu + i\theta\sigma^\mu\bar{\varepsilon} - i\varepsilon\sigma^\mu\bar{\theta},\ \theta+\varepsilon,\ \bar{\theta}+\bar{\varepsilon}).$$

This transformation may be generated by the differential operators Q and $\bar{Q}$ (we use the same name as the susy generators):

$$i\varepsilon Q + i\bar{\varepsilon}\bar{Q} = i\varepsilon^\alpha\left(-i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\right) + i\left(i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu\right)\bar{\varepsilon}^{\dot{\alpha}},$$

so that we can identify the superspace representation of the susy generators

$$\begin{cases} Q_\alpha = -i\dfrac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{Q}_{\dot{\alpha}} = i\dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \end{cases}$$

One can easily check (with the help of the definition of the derivatives on the θ and $\bar{\theta}$ coordinates which will be given below) that Q_α and $\bar{Q}_{\dot{\alpha}}$ satisfy the susy algebra

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu = -2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu .$$

It is useful to introduce some "covariant derivatives" which anticommute with the susy generators:

$$\begin{cases} D_\alpha = \dfrac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{D}_{\dot{\alpha}} = \dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \end{cases}$$

where $\bar{D}_{\dot{\alpha}} = (D_\alpha)^\dagger$. One can check the following anticommutation rules

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu, \qquad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0,$$

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = 0 .$$

A superfield is defined as a field on the superspace:

$$F(x,\theta,\bar{\theta}).$$

We can find the action of the susy generators on the superfield F for infinitesimal transformations

$$(1 + i\varepsilon Q + i\bar{\varepsilon}\bar{Q})F(x^\mu,\theta^\alpha,\bar{\theta}^{\dot{\beta}}) = F(x^\mu - i\varepsilon\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\varepsilon},\ \theta^\alpha + \varepsilon^\alpha,\ \bar{\theta}^{\dot{\beta}} + \bar{\varepsilon}^{\dot{\beta}})$$

and the susy variation of a superfield is defined by

$$\delta_{\varepsilon,\bar{\varepsilon}}F = (i\varepsilon Q + i\bar{\varepsilon}\bar{Q})F .$$

The linearity of the representation of the susy generators Q_α and $\bar{Q}_{\dot\alpha}$ has the consequence that the sum and the product of two superfields are still superfields:

$$\delta(F_1+F_2)=(i\epsilon Q+i\bar{\epsilon}\bar{Q})(F_1+F_2)=(i\epsilon Q+i\bar{\epsilon}\bar{Q})F_1+(i\epsilon Q+i\bar{\epsilon}\bar{Q})F_2=\delta F_1+\delta F_2$$

and

$$\delta(F_1\cdot F_2)=(i\epsilon Q+i\bar{\epsilon}\bar{Q})(F_1\cdot F_2)=[(i\epsilon Q+i\bar{\epsilon}\bar{Q})F_1]\cdot F_2+F_1\cdot[(i\epsilon Q+i\bar{\epsilon}\bar{Q})F_2]=\delta F_1\cdot F_2+F_1\cdot\delta F_2.$$

NOTE. Usually we call superfields only the functions on the superspace which carry a representation of the susy algebra

- Notations and conventions

Now we fix the notations and conventions which are needed to properly define the N=1 rigid superspace. The superspace coordinates anticommute

$$\{\theta_\alpha,\theta_\beta\}=\{\bar\theta_{\dot\alpha},\bar\theta_{\dot\beta}\}=\{\theta_\alpha,\bar\theta_{\dot\beta}\}=0.$$

Spinor indices are contracted in the usual way

$$\theta\theta\equiv\theta^\alpha\theta_\alpha$$
$$\bar\theta\bar\theta\equiv\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$$

One can prove the following identities

$$\theta^\alpha\theta^\beta=-\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta\ ,\qquad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}=\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta\ ,$$

$$\theta_\alpha\theta_\beta=\frac{1}{2}\epsilon_{\alpha\beta}\theta\theta\ ,\qquad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}=-\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta\ ,$$

$$(\theta\psi)(\theta\chi)=-\frac{1}{2}(\theta\theta)(\psi\chi)\ ,\qquad (\bar\theta\bar\psi)(\bar\theta\bar\chi)=-\frac{1}{2}(\bar\theta\bar\theta)(\bar\psi\bar\chi)\ ,$$

$$(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)=\frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\eta^{\mu\nu}\ .$$

Derivatives in θ and $\bar\theta$ are defined as

$$\partial_\alpha\theta^\beta=\frac{\partial}{\partial\theta^\alpha}\theta^\beta=\delta_\alpha^\beta\ ,\qquad \partial^\alpha\equiv\frac{\partial}{\partial\theta_\alpha}=-\epsilon^{\alpha\beta}\partial_\beta\ ,$$

$$\bar\partial_{\dot\alpha}\bar\theta^{\dot\beta}=\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\theta^{\dot\beta}=\delta_{\dot\alpha}^{\dot\beta}\ ,\qquad \bar\partial^{\dot\alpha}\equiv\frac{\partial}{\partial\bar\theta_{\dot\alpha}}=-\epsilon^{\dot\alpha\dot\beta}\bar\partial_{\dot\beta}\ .$$

Superspace derivatives act as left-derivatives, that is we must move to the leftmost position the variable we are differentiating; for example:

$$\frac{\partial}{\partial\theta^\alpha}\xi^\beta\theta^\gamma=-\frac{\partial}{\partial\theta^\alpha}\theta^\gamma\xi^\beta=-\delta_\alpha^\gamma\xi^\beta.$$

We can prove the relations

$$\partial_\alpha(\theta\theta)=2\theta_\alpha\ ,\qquad \bar\partial_{\dot\alpha}(\bar\theta\bar\theta)=-2\bar\theta_{\dot\alpha}\ ,$$

$$\partial^2(\theta\theta)\equiv\partial^\alpha\partial_\alpha(\theta\theta)=4\ ,\qquad \bar\partial^2(\bar\theta\bar\theta)\equiv\bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha}(\bar\theta\bar\theta)=4\ .$$

From the above definitions we get $(\partial_\alpha)^+=\bar\partial_{\dot\alpha}$ with $\alpha\equiv\dot\alpha$, notice the + sign rather than the − sign which appears in the relation $(\partial_\mu)^+=-\partial_\mu$.

We can also introduce an integration on the superspace. It is defined as the Berezin integral for Grassmann variables η:

$$\int d\eta\ \eta = 1 \quad , \quad \int d\eta\ 1 = 0$$

$$\int d\eta\ f(\eta) = \int d\eta\ (f_0 + \eta f_1) = f_1 .$$

Berezin integral is translationally invariant

$$\int d(\eta+\xi)\ f(\eta+\xi) = \int d\eta\ f(\eta)$$

$$\int d\eta \frac{\partial}{\partial\eta} f(\eta) = 0 \quad ,$$

it is equivalent to the differentiation

$$\frac{d}{d\eta} f(\eta) = f_1 = \int d\eta\ f(\eta) ,$$

and we can also define a Grassmann delta function

$$\delta(\eta) \equiv \eta .$$

<u>NOTE</u>. Given that $\eta^2 = 0$ we have a finite series expansion in η: $f(\eta) = f_0 + \eta f_1$.

These properties are easily generalized to the superspace coordinates $\theta^\alpha, \bar\theta^{\dot\alpha}$. In this case we must pay attention to the order of integration variables: we integrate bringing the variable to the leftmost position and the first integration is the innermost one:

$$\int d\theta^1 d\theta^2\ \theta^2\theta^1 \equiv \int d\theta^1 \left(\int d\theta^2\ \theta^2\right)\theta^1 = \int d\theta^1\ \theta^1 = 1 ,$$

$$\int d\theta^2 d\theta^1\ \theta^2\theta^1 = -\int d\theta^2 d\theta^1\ \theta^1\theta^2 = -\int d\theta^1 d\theta^2\ \theta^2\theta^1 = -1 .$$

Since

$$\theta\theta = 2\theta^2\theta^1 \quad , \quad \bar\theta\bar\theta = 2\bar\theta^{\dot 1}\bar\theta^{\dot 2}$$

we define

$$d^2\theta \equiv \frac{1}{2} d\theta^1 d\theta^2 = -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \quad ; \quad d^2\bar\theta \equiv \frac{1}{2} d\bar\theta^{\dot 2} d\bar\theta^{\dot 1} = -\frac{1}{4} d\bar\theta_{\dot\alpha} d\bar\theta_{\dot\beta} \epsilon^{\dot\alpha\dot\beta} = [d^2\theta]^+$$

so that

$$\int d^2\theta\ (\theta\theta) = \int d^2\bar\theta\ (\bar\theta\bar\theta) = 1 .$$

It is easy to check that

$$\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} \quad , \quad \int d^2\bar\theta = -\frac{1}{4}\epsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta^{\dot\beta}} ,$$

and

$$\int d^4\theta (\theta\theta)(\bar\theta\bar\theta) \equiv \int d^2\theta\ d^2\bar\theta\ (\theta\theta)(\bar\theta\bar\theta) = 1 .$$

· The general N=1 scalar superfield

Now we can construct the most general N=1 scalar superfield. We recall that the components of θ and $\bar\theta$ anticommute, so any product of these components vanishes if two of them are the same component. In total θ and $\bar\theta$ have only 4 components, so any function of θ and $\bar\theta$ has a power series that terminate with its quartic term.

The most generic component expansion of the superfield is

$$\Phi(x,\theta,\bar\theta) = f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) + \theta\theta\, m(x) + \bar\theta\bar\theta\, n(x)$$
$$+\theta\sigma^\mu\bar\theta\, v_\mu(x) + (\theta\theta)\,\bar\theta\bar\lambda(x) + (\bar\theta\bar\theta)\theta\,\psi(x) + (\theta\theta)(\bar\theta\bar\theta)\, d(x).$$

Notice that if $\Phi(x,\theta,\bar\theta)$ is a scalar, as we chose before, then $f(x)$, $m(x)$, $n(x)$, $v_\mu(x)$ and $d(x)$ are bosonic fields, while $\phi(x)$, $\bar\chi(x)$, $\bar\lambda(x)$ and $\psi(x)$ are fermionic fields.

To obtain the above expansion for Φ we used the spinor identities to remove some redundant terms like $\bar\theta\bar\sigma^\mu\theta v_\mu$.

We can compute the variation of the superfield components under a susy transformation:

$$\delta_\varepsilon\Phi(x,\theta,\bar\theta) = (i\varepsilon Q + i\bar\varepsilon\bar Q)\,\Phi(x,\theta,\bar\theta)$$
$$= \left(\varepsilon^\alpha\frac{\partial}{\partial\theta^\alpha} - i\varepsilon^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu - \frac{\partial}{\partial\bar\theta^{\dot\alpha}}\bar\varepsilon^{\dot\alpha} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\bar\varepsilon^{\dot\alpha}\partial_\mu\right)\Phi(x,\theta,\bar\theta)$$
$$= \varepsilon\phi + \bar\varepsilon\bar\chi + i\theta\sigma^\mu\bar\varepsilon\,\partial_\mu f + 2\varepsilon\theta\, m + \theta\sigma^\mu\bar\varepsilon\, v_\mu - i\varepsilon\sigma^\mu\bar\theta\,\partial_\mu f$$
$$+ 2\bar\varepsilon\bar\theta\, n + \varepsilon\sigma^\mu\bar\theta\, v_\mu + i(\theta\sigma^\mu\bar\varepsilon)\theta\partial_\mu\phi + (\theta\theta)(\bar\varepsilon\bar\lambda) - i(\varepsilon\sigma^\mu\bar\theta)\bar\theta\partial_\mu\bar\chi$$
$$+ (\bar\theta\bar\theta)(\varepsilon\psi) - i(\varepsilon\sigma^\mu\bar\theta)\theta\partial_\mu\phi + i(\theta\sigma^\mu\bar\varepsilon)\bar\theta\partial_\mu\bar\chi + 2(\varepsilon\theta)(\bar\theta\bar\lambda) + 2(\bar\varepsilon\bar\theta)(\theta\psi)$$
$$- i(\varepsilon\sigma^\mu\bar\theta)(\theta\theta)\partial_\mu m + i(\theta\sigma^\mu\bar\varepsilon)(\theta\sigma^\nu\bar\theta)\partial_\mu v_\nu + 2(\theta\theta)(\bar\varepsilon\bar\theta)d$$
$$+ i(\theta\sigma^\mu\bar\varepsilon)(\bar\theta\bar\theta)\partial_\mu n - i(\varepsilon\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)\partial_\mu v_\nu + 2(\varepsilon\theta)(\bar\theta\bar\theta)d$$
$$- i(\varepsilon\sigma^\mu\bar\theta)(\theta\theta)\bar\theta\partial_\mu\bar\lambda + i(\theta\sigma^\mu\bar\varepsilon)(\bar\theta\bar\theta)\theta\partial_\mu\psi.$$

Using the Fierz identities we find the component fields transformation rules

$$\delta_\varepsilon f = \varepsilon\phi + \bar\varepsilon\bar\chi$$
$$\delta_\varepsilon\phi_\alpha = 2\varepsilon_\alpha m + \sigma^\mu_{\alpha\dot\beta}\bar\varepsilon^{\dot\beta}(i\partial_\mu f + v_\mu)$$
$$\delta_\varepsilon\bar\chi^{\dot\alpha} = 2\bar\varepsilon^{\dot\alpha} n + \varepsilon^\beta\sigma^\mu_{\beta\dot\gamma}\,\varepsilon^{\dot\gamma\dot\alpha}(i\partial_\mu f - v_\mu)$$
$$\delta_\varepsilon m = \bar\varepsilon\bar\lambda - \frac{i}{2}\partial_\mu\phi\,\sigma^\mu\bar\varepsilon$$
$$\delta_\varepsilon n = \varepsilon\psi + \frac{i}{2}\varepsilon\sigma^\mu\partial_\mu\bar\chi$$

$$\delta_\varepsilon v_\mu = \varepsilon\sigma_\mu\bar\lambda + \psi\sigma_\mu\bar\varepsilon + \frac{i}{2}\partial_\nu\phi\,\sigma_\mu\bar\sigma^\nu\varepsilon - \frac{i}{2}\partial_\nu\bar\chi\,\bar\sigma_\mu\sigma^\nu\bar\varepsilon$$

$$\delta_\varepsilon\bar\lambda^{\dot\alpha} = 2\bar\varepsilon^{\dot\alpha}d + i\,\bar\sigma^{\mu\dot\alpha\beta}\varepsilon_\beta\partial_\mu m - \frac{i}{2}\bar\sigma^{\nu\dot\alpha\beta}\sigma^\mu{}_{\beta\dot\gamma}\bar\varepsilon^{\dot\gamma}\partial_\mu v_\nu$$

$$\delta_\varepsilon\psi_\alpha = 2\varepsilon_\alpha d + i\,\sigma^\mu{}_{\alpha\dot\beta}\bar\varepsilon^{\dot\beta}\partial_\mu n - \frac{i}{2}\sigma^\nu{}_{\alpha\dot\beta}\bar\sigma^{\mu\dot\beta\gamma}\varepsilon_\gamma\partial_\mu v_\nu$$

$$\delta_\varepsilon d = \frac{i}{2}\partial_\mu\left(-\psi\sigma^\mu\bar\varepsilon + \varepsilon\sigma^\mu\bar\lambda\right)$$

An important property of the above transformation is the fact that d transforms as a total derivative. This will be useful to construct supersymmetric actions.

The above transformation rules show that the general scalar superfield forms a basis for an (off-shell) linear representation of $N=1$ supersymmetry. However this representation is <u>reducible</u>. For example we could impose the constraints

$$\begin{cases} \chi = 0 \\ n = 0 \\ v_\mu = \partial_\mu f \\ \bar\lambda = \frac{i}{2}\partial_\mu\phi\,\sigma^\mu \\ \psi = 0 \\ d = -\frac{1}{4}\Box f \end{cases}$$

one can easily check that the transformations of the fields under susy respect these constraints, so they define a susy representation (actually they define an irreducible representation).

Of course, it is not practical to guess all possible constraints which can give a susy representation from the general scalar superfield. In the following we will see how we can use some operators to set some susy-invariant constraints on the general superfield to get irreducible representations.

Notice that not any operation, which we apply on a superfield, gives another superfield. For a function on the superspace to be a superfield which carry a susy representation we must require that its form is not modified by susy transformations.

For example summing or multiplying two superfields gives a superfield. In the same way a spacetime derivative of a superfield is still a superfield. On the other hand by taking a derivative with respect to the fermionic superspace coordinates (∂_α or $\bar\partial_{\dot\alpha}$) or by multiplying a field with θ or $\bar\theta$ we do not get a superfield which carry a susy representation because we eliminate some component fields in a non-susy-invariant way.