V. SUPERSYMMETRY REPRESENTATIONS ON QUANTUM FIELDS

So far we have only seen representations of SUSY on physical on-shell states. However, in order to build supersymmetric field theories, we also need to find the supersymmetry representations on quantum fields. As we will see in the following representations on fields will be in general off-shell representations, hence they contain some unphysical degrees of freedom, which are nevertheless necessary for treating a quantum field theory at the radiative level. The extra degrees of freedom are, of course, removed when we require the fields to be on-shell, that is when the equations of motion are imposed.

The by far simplest field representations of the susy algebra are the ones for simple supersymmetry ($N=1$). In the following we will explicitly construct the simplest field representation of $N=1$ susy, the so called "chiral superfield".

As a first step to understand supersymmetry in field theory, we will construct the chiral superfield by brute force, then we will learn a more convenient technique, the "superspace" formalism, which allows to find easily $N=1$ susy representations and to build supersymmetric field theories.

· Direct construction of field supermultiplets

We want to build a field which corresponds to the simplest supermultiplet of arbitrary mass in $N=1$. To build the multiplet we start from a scalar field which creates the spin zero states of the multiplet from the vacuum. To reproduce the condition that the Clifford vacuum is annihilated by a subset of the susy generators we impose our scalar field $\phi(x)$ to commute with the $\bar{Q}_{\dot\alpha}$ operators

$$[\bar{Q}_{\dot\alpha}, \phi(x)] = 0 . \qquad (*)$$

When the field $\phi(x)$ is applied to a susy invariant vacuum $|0\rangle$ it creates states which are annihilated by the $\bar{Q}_{\dot\alpha}$. (The choice of the $\bar{Q}_{\dot\alpha}$ is arbitrary, we could have chosen the same relation with Q_α instead of $\bar{Q}_{\dot\alpha}$.)

In order to build a non-trivial representation we must assume that $\phi(x)$ is a complex field. If it is real then by taking the conjugate of relation $(*)$ we would get

$$[Q_\alpha, \phi(x)] = 0 \quad \Leftarrow \quad \text{if } \phi^*(x) = \phi(x)$$

but then

$$0 = [\{Q_\alpha, \bar{Q}_{\dot\alpha}\}, \phi(x)] = 2\sigma^\mu_{\alpha\dot\alpha}[P_\mu, \phi(x)] = -2\sigma^\mu_{\alpha\dot\alpha}\, i\,\partial_\mu\phi(x) \qquad (**)$$

which would imply $\phi(x) = \text{const}$. (We use the representation $P_\mu \equiv -i\partial_\mu$ on fields.)

We thus conclude that

$$[Q_\alpha, \phi(x)] \neq 0.$$

NOTE. Equation (**) has also another consequence: it prohibits trivial representations of the susy algebra on fields. If we assume that a field ϕ is invariant under susy transformations, it would commute with Q_α and $\bar{Q}_{\dot\alpha}$, thus it should be constant: $\phi = \text{const}$.

To build a representation we must find a set of fields which transform one into each other under the action of a susy transformation. A simple strategy to do this is to use the commutators of the fields with the susy generators in order to obtain the various components of the multiplet. We continue to compute commutators until the representation is complete, that is until the set of fields we find remains in itself under a generic susy transformation.

We start from the commutator of $\phi(x)$ with Q_α; it is a fermionic field which we parametrize as

$$\underline{[\phi(x), Q_\alpha] \equiv \sqrt{2}\, i\, \psi_\alpha}.$$

We can then apply the susy transformations to ψ_α:

$$\begin{cases} \{\psi_\alpha, Q_\beta\} \equiv -i\sqrt{2}\, F_{\alpha\beta} \\ \{\psi_\alpha, \bar{Q}_{\dot\beta}\} \equiv X_{\alpha\dot\beta} \end{cases}$$

We can now use the susy algebra to find some constraints on $F_{\alpha\beta}$ and $X_{\alpha\dot\beta}$. From the Jacobi identity with ϕ, Q_α and $\bar{Q}_{\dot\beta}$ we get

$$0 = \{[\phi(x), Q_\alpha], \bar{Q}_{\dot\beta}\} + [\{Q_\alpha, \bar{Q}_{\dot\beta}\}, \phi(x)] - \{\cancel{[\bar{Q}_{\dot\beta}, \phi(x)]}, Q_\alpha\}$$

$$= \{\sqrt{2}\, i\, \psi_\alpha, \bar{Q}_{\dot\beta}\} + [2\sigma^\mu_{\alpha\dot\beta} P_\mu, \phi(x)]$$

$$= \sqrt{2}\, i\, X_{\alpha\dot\beta} - 2i\sigma^\mu_{\alpha\dot\beta}\, \partial_\mu \phi(x)$$

which implies

$$\underline{X_{\alpha\dot\beta} = \sqrt{2}\, \sigma^\mu_{\alpha\dot\beta}\, \partial_\mu \phi(x)}.$$

This means that $X_{\alpha\dot\beta}$ is <u>not</u> a new degree of freedom, because it can be expressed in terms of the $\phi(x)$ field.

We now consider the Jacobi identity with ϕ, Q_α and Q_β:

$$0 = [\{Q_\alpha, \cancel{Q_\beta}\}, \phi(x)] - \{[Q_\beta, \phi(x)], Q_\alpha\} + \{[\phi(x), Q_\alpha], Q_\beta\}$$

$$= +\sqrt{2}i\,\{\psi_\beta, Q_\alpha\} + \sqrt{2}i\,\{\psi_\alpha, Q_\beta\}$$

$$= 2F_{\beta\alpha} + 2F_{\alpha\beta}\,.$$

This implies that $F_{\alpha\beta}$ is antisymmetric in α and β, so that

$$F_{\alpha\beta} = -F_{\beta\alpha} \quad \Rightarrow \quad \underline{F_{\alpha\beta} = \varepsilon_{\alpha\beta} F}\,,$$

where $F(x)$ is a complex scalar field.

We now compute the action of Q_α and $\bar{Q}_{\dot\alpha}$ on F; we define

$$\begin{cases} [F, Q_\alpha] \equiv \lambda_\alpha \\ [F, \bar{Q}_{\dot\alpha}] \equiv \bar{\chi}_{\dot\alpha} \end{cases}$$

By using the Jacobi identity with ψ_α, Q_β and $\bar{Q}_{\dot\beta}$ we get

$$0 = [\{\psi_\alpha, Q_\beta\}, \bar{Q}_{\dot\beta}] + [\{Q_\beta, \bar{Q}_{\dot\beta}\}, \psi_\alpha] + [\{\bar{Q}_{\dot\beta}, \psi_\alpha\}, Q_\beta]$$

$$= -\sqrt{2}i\,\varepsilon_{\alpha\beta}[F, \bar{Q}_{\dot\beta}] - 2i\,\sigma^\mu_{\beta\dot\beta}\,\partial_\mu\psi_\alpha + [X_{\alpha\dot\beta}, Q_\beta]$$

$$= -\sqrt{2}i\,\varepsilon_{\alpha\beta}\,\bar{\chi}_{\dot\beta} - 2i\,\sigma^\mu_{\beta\dot\beta}\,\partial_\mu\psi_\alpha + \sigma^\mu_{\alpha\dot\beta}\cdot 2i\,\partial_\mu\psi_\beta\,.$$

By contracting with $\varepsilon^{\beta\alpha}$ we get

$$-2\sqrt{2}i\,\bar{\chi}_{\dot\beta} - 2i\,\partial_\mu\psi^\beta\sigma^\mu_{\beta\dot\beta} - 2i\,\partial_\mu\psi^\alpha\sigma^\mu_{\alpha\dot\beta} = 0$$

which implies that $\bar{\chi}_{\dot\beta}$ can be expressed in terms of $\partial_\mu\psi_\alpha$:

$$\underline{\bar{\chi}_{\dot\beta} = -\sqrt{2}\,\partial_\mu\psi^\beta\sigma^\mu_{\beta\dot\beta}}\,.$$

We can show that λ_α vanishes by using the Jacobi identity

$$0 = [\{\psi_\alpha, Q_\beta\}, Q_\gamma] + [\{Q_\beta, \cancel{Q_\gamma}\}, \psi_\alpha] + [\{Q_\gamma, \psi_\alpha\}, Q_\beta]$$

$$= -i\,\varepsilon_{\alpha\beta}[F, Q_\gamma] - i\,\varepsilon_{\alpha\gamma}[F, Q_\beta]$$

$$= -i\,\varepsilon_{\alpha\beta}\lambda_\gamma - i\,\varepsilon_{\alpha\gamma}\lambda_\beta\,.$$

By choosing $\alpha = \gamma \neq \beta$ we get

$$\underline{\lambda_\alpha = 0}\,.$$

This shows that the set of fields (ϕ, ψ_α, F) form a representation of the $N=1$ supersymmetry algebra. This multiplet is called <u>"chiral multiplet"</u>.

By taking the adjoint of the above multiplet we get the <u>"antichiral multiplet"</u> with components $(\phi^+, \bar\psi_{\dot\alpha}, F^+)$. This multiplet can also be obtained by starting from the condition $[\bar{Q}_{\dot\alpha}, \phi^+] = 0$.

Let's count the degrees of freedom of the chiral multiplet. There are a total of 4 bosonic degrees of freedom ($\mathrm{Re}\,\phi$, $\mathrm{Im}\,\phi$, $\mathrm{Re}\,F$, $\mathrm{Im}\,F$) and 4 fermionic degrees of freedom ($\mathrm{Re}\,\psi_1$, $\mathrm{Im}\,\psi_1$, $\mathrm{Re}\,\psi_2$, $\mathrm{Im}\,\psi_2$). The multiplet has 4+4 degrees of freedom. This is the smallest possible number in 4 space-time dimension, since any multiplet must contain a spinor and spinors have at least two complex (Weyl) or four real components (Majorana). The multiplet is thus irreducible. Notice that this counting gives the off-shell degrees of freedom. To find the on-shell degrees of freedom we need to take into account the constraints given by the equations of motion.

Usually one introduces anticommuting parameters ζ^α and $\bar\zeta^{\dot\alpha}$ which are defined to anticommute with everything fermionic (including themselves) and to commute with everything bosonic (including ordinary c-numbers). Mathematically speaking, they are given by Grassmann variables.

With the help of the spinor parameters we can define infinitesimal variations of a field X by

$$\underline{\delta X \equiv [\zeta Q + \bar Q \bar\zeta, X]}$$

Notice that the ζQ and $\bar Q\bar\zeta$ objects are now commuting objects, so they require commutation relations.

For the chiral multiplet we get

$$\begin{cases} \delta\phi = -\sqrt{2}\, i\,\zeta\psi \\ \delta\psi = -i\sqrt{2}\, F\zeta - \sqrt{2}\,\partial_\mu\phi\,\sigma^\mu\bar\zeta \\ \delta F = \sqrt{2}\,\partial_\mu\psi\,\sigma^\mu\bar\zeta \end{cases} .$$

We can explicitly check that the above transformations satisfy the algebra. We compute the commutator of two transformations

$$\begin{aligned}(\delta_1\delta_2-\delta_2\delta_1)X &= [\zeta_1 Q+\bar\zeta_1\bar Q, [\zeta_2 Q+\bar\zeta_2\bar Q, X]] - [\zeta_2 Q+\bar\zeta_2\bar Q, [\zeta_1 Q+\bar\zeta_1\bar Q, X]] \\ &= [\zeta_1 Q+\bar\zeta_1\bar Q, [\zeta_2 Q+\bar\zeta_2\bar Q, X]] + [\zeta_2 Q+\bar\zeta_2\bar Q, [X, \zeta_1 Q+\bar\zeta_1\bar Q]] \\ &= -[X, [\zeta_1 Q+\bar\zeta_1\bar Q, \zeta_2 Q+\bar\zeta_2\bar Q]] \\ &= [X, +2i(\zeta_1\sigma^\mu\bar\zeta_2 - \zeta_2\sigma^\mu\bar\zeta_1)\partial_\mu] \\ &= -2i(\zeta_1\sigma^\mu\bar\zeta_2 - \zeta_2\sigma^\mu\bar\zeta_1)\partial_\mu X ,\end{aligned}$$

and we can check that the above transformation rules satisfy the algebra.

With the chiral multiplet we can build a single supersymmetric Lagrangian: the Wess-Zumino Lagrangian

$$\mathcal{L}_{WZ} = \partial_\mu \phi^\dagger \partial^\mu \phi - i\bar\psi\bar\sigma^\mu \partial_\mu \psi + F^\dagger F .$$

This is the simplest susy Lagrangian and contains only kinetic terms for a scalar and a fermion field.

One can show that $\mathcal{L}_{WZ}$ gives an action which is invariant under susy transformations

$$\mathcal{L}_{WZ} \to \mathcal{L}_{WZ} + \partial_\mu \xi^\mu \Rightarrow \quad S_{WZ} \equiv \int d^4x\, \mathcal{L}_{WZ} \quad \text{is invariant.}$$

The action contains also an auxiliary field F, which is a non-dynamical field, that is, it has no kinetic term in the Lagrangian.

We can now discuss the set of on-shell degrees of freedom. The equations of motion are

$$\begin{cases} \partial_\mu \partial^\mu \phi = 0 \\ i\bar\sigma^\mu \partial_\mu \psi = 0 \\ F = 0 \end{cases} .$$

The equations of motion eliminate the F field and two real degrees of freedom from ψ. Thus the on-shell degrees of freedom are

$$\begin{cases} 2 \text{ bosonic} \\ 2 \text{ fermionic.} \end{cases}$$

We can also use the $F=0$ equation to simplify the Lagrangian

$$\tilde{\mathcal{L}}_{WZ} = \partial_\mu \phi^\dagger \partial^\mu \phi - i\psi\sigma^\mu \partial_\mu \bar\psi ,$$

at the same time the transformation rules for the fields become

$$\begin{cases} \delta\phi = -\sqrt{2}\, i\,\zeta\psi \\ \delta\psi = -\sqrt{2}\,\partial_\mu \phi\, \sigma^\mu \bar\zeta \end{cases}$$

In this case the transformation rules satisfy the susy algebra only if we also impose the equations of motion:

$$\begin{aligned}[\delta_1, \delta_2]\psi &= -2i(\zeta_1\sigma^\mu\bar\zeta_2 - \zeta_2\sigma^\mu\bar\zeta_1)\partial_\mu\psi - 2i\left[(\partial_\mu\psi\,\sigma^\mu\bar\zeta_2)\zeta_1 - (\partial_\mu\psi\,\sigma^\mu\bar\zeta_1)\zeta_2\right] \\ &= -2i(\zeta_1\sigma^\mu\bar\zeta_2 - \zeta_2\sigma^\mu\bar\zeta_1)\partial_\mu\psi + 2i\left[(\zeta_2\bar\sigma^\mu\partial_\mu\bar\psi)\zeta_1 - (\zeta_1\bar\sigma^\mu\partial_\mu\bar\psi)\zeta_2\right].\end{aligned}$$

To find the above equation we need the Fierz identity $\chi_\alpha(\xi\eta) = -\xi_\alpha(\eta\chi) - \eta_\alpha(\xi\chi)$ and the relation $\bar\xi\bar\sigma^\mu\chi = -\chi\sigma^\mu\bar\xi$. The equations of motion eliminate the terms in square brackets, thus leaving only the terms needed to satisfy the susy algebra.